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PREFACE.

This volume contains solutions of nearly all the examples found in Robinson's New University Algebra. The greater part of these solutions might have been omitted, if the sole object of a Key were to aid teachers, of limited acquirements and experience, in overcoming difficulties otherwise insurmountable by them. Though this may be one of the primary objects of a Key, there is, nevertheless, a higher and far more important use.

This higher object is purely educational, and is simply an extension of the author's design in the solutions given in the text-book. It is to illustrate more completely the principles of the science, in their practical bearings; and to show the application of the rules and methods taught, under the greatest possible variety of circumstances. Viewed in this light, and used with discretion, a Key to an Algebra may be serviceable to all students of the science, and especially to teachers who, by reason of limited educational advantages, find it necessary to apply themselves diligently to self-culture in their profession.

These remarks apply more particularly to those examples which are designed to tax the ingenuity, and to call forth algebraic skill. In the case of examples requiring long and

tedious numerical calculations, a Key is often a great convenience in detecting mistakes, the correction of which would require from the teacher, under the pressure of school-room duties, a needless expenditure of time and labor.

We have given full solutions of the numerical equations of higher degrees, found on the last page of the Algebra,—a feature of the work which will be acceptable to all who use the book. The attention of teachers is invited particularly to the system of decimal contraction applied in these solutions, and to the convenient method of marking the corresponding terms in the several columns.

SEPTEMBER, 1862.

KEY TO

ROBINSON'S

NEW UNIVERSITY ALGEBRA.

USE OF THE PARENTHESIS.

(**61**, page 29.)

1. $3a+(2b^2-a-d+m)=3a+2b^2-a-d+m$
$$=2a+2b^2-d+m, \textit{ Ans.}$$

2. $4x^2-y-(3x-7y+5)+2x=4x^2-y-3x+7y-5+2x$
$$=4x^2+6y-x-5, \textit{ Ans.}$$

3. $a+2c-(4c-3a+2m^2)=a+2c-4c+3a-2m^2$
$$=4a-2c-2m^2, \textit{ Ans.}$$

4. $4x^3-2x^2-[x^3-(2x^2+5x-7)-6x+1]=4x^3-2x^2-x^3+(2x^2+5x-7)+6x-1=4x^3-2x^2-x^3+2x^2+5x-7+6x-1=$
$$3x^3+11x-8, \textit{ Ans.}$$

5. $a+2m-\{c+x-[a-m-(c-2x)]\}$
$=a+2m-c-x+[a-m-(c-2x)]$
$=a+2m-c-x+a-m-c+2x=2a+m-2c+x$, *Ans.*

6. $3x^2-4x-am-\{x^2-x-[3am-(2x+2am)+2x^2]-5am\}$
$=3x^2-4x-am-x^2+x+[3am-(2x+2am)+2x^2]+5am$
$=3x^2-4x-am-x^2+x+3am-2x-2am+2x^2+5am$
$$=4x^2-5x+5am, \textit{ Ans.}$$

7. $3a-\{2m^2+[5c-9a-(3u+m^2)]+6a-(m^2+5c)\}$
$=3a-2m^2-[5c-9a-(3a+m^2)]-6a+m^2+5c$
$=3a-2m^2-5c+9a+3a+m^2-6a+m^2+5c=9a$, *Ans.*

8. $x^2-\{5mc^2-[x^2-(3c-3mc^2)+3c-(x^2-2mc^2-c)]\}$
$=x^2-5mc^2+[x^2-(3c-3mc^2)+3c-(x^2-2mc^2-c)]$
$=x^2-5mc^2+x^2-3c+3mc^2+3c-x^2+2mc^2+c$
$=x^2+c$, *Ans.*

9. $m^2-m-1-\{m^2-2m-2-[m^2-3m-3-(m^2-4m-4)]\}$
$=m^2-m-1-m^2+2m+2+[m^2-3m-3-(m^2-4m-4)]$
$=m^2-m-1-m^2+2m+2+m^2-3m-3-m^2+4m+4$
$=2m+2$, *Ans.*

10. $5z^3-3z^2+4z-1-[2z^3-(3z^2-2z+1)-z^2+z]$
$=5z^3-3z^2+4z-1-2z^3+3z^2-2z+1+z^2-z$
$=4z^2+z$, *Ans.*

11. $4c^3-2c^2+c+1-(3c^3-c^2-c-7)-(c^3-4c^2+2c+8)$
$=4c^3-2c^2+c+1-3c^3+c^2+c+7-c^3+4c^2-2c-8$
$=3c^2$, *Ans.*

12. $3a^2b-4cd-(3cd-2a^2b)-[a^3+c-(5cd+3a^2b)+(3a^3+2cd)+a^3]=3a^2b-4cd-3cd+2a^2b-a^3-c+(5cd+3a^2b)-(3a^3+2cd)-a^3=3a^2b-4cd-3cd+2a^2b-a^3-c+5cd+3a^2b-3a^3-2cd-a^3$
$=8a^2b-4cd-5a^3-c$, *Ans.*

13. $-\left(4a^2m+3m^2d-(7m^2d-9a^2m-n)-\{5n-[m^2d-(2n+a^2m)+3an^2]-5a^2m\}-12a^2m\right)=-4a^2m-3m^2d+(7m^2d-9a^2m-n)+\{5n-[m^2d-(2n+a^2m)+3an^2]-5a^2m\}+12a^2m=-4a^2m-3m^2d+7m^2d-9a^2m-n+5n-[m^2d-(2n+a^2m)+3a^2n]-5a^2m+12a^2m=-4a^2m-3m^2d+7m^2d-9a^2m-n+5n-m^2d+2n+a^2m-3an^2-5a^2m+12a^2m$
$=3m^2d+6n-5a^2m-3an^2$, *Ans.*

FACTORING.

(95, page 51.)

5. $x^3-x^2y+xy^2-y^3=x^2(x-y)+y^2(x-y)=(x^2+y^2)(x-y)$, *Ans.*

6. $a^4b^2+2a^3b^3+a^2b^4=a^2b^2(a^2+2ab+b^2)$
$=a^2b^2(a+b)(a+b)$, *Ans.*

7. $(x^2-x)a+(x^2+x)(3b-c)-q = ax^2-ax+3bx^2+3bx-cx^2-cx-q = ax^2+3bx^2-cx^2-ax+3bx-cx-q$
$$=(a+3b-c)x^2-(a-3b+c)x-q, \textit{ Ans.}$$

8. $a^5m-9am^3=am(a^4-9m^2)=am(a^2+3m)(a^2-3m)$, *Ans.*

9. By (**89**, 4), $8a^3-x^3=(2a-x)(4a^2+2ax+x^2)$, *Ans.*

10. y^5+243 is the sum of the fifth powers of y, and 3; hence by (**89**, 1),
$$y^5+243=(y+3)(y^4-3y^3+9y^2-27y+81), \textit{ Ans.}$$

11. By (**70**, III.), we have
$$x^6-y^6=(x^3+y^3)(x^3-y^3).$$
And by (**89**, 1 and 4),
$$(x^3+y^3)(x^3-y^3)=(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2), \textit{ Ans.}$$

12. $$a^3-ab^2+2abc-ac^2 = a(a^2-b^2+2bc-c^2)$$
$$= a\{a^2-(b-c)^2\}$$
Or by (**70**, III.), $$= a(a+b-c)(a-b+c), \textit{ Ans.}$$

SUBSTITUTION.

(**96**, page 51.)

1.
$$\begin{array}{rcl} a^2=(a-b)^2 & = & a^2-2ab+b^2 \\ ab=(a-b)b & = & ab-b^2 \\ b^2 & = & b^2 \\ \hline a^2+ab+b^2 & = & a^2-ab+b^2, \textit{ Ans.} \end{array}$$

2.
$$\begin{array}{rcl} a^2=(x+2)^2 & = & x^2+4x+4 \\ -2a=-2(x+2) & = & -2x-4 \\ 1 & = & 1 \\ \hline a^2-2a+1 & = & x^2+2x+1, \textit{ Ans.} \end{array}$$

3.
$$\begin{array}{rcl} y^4=(x+3)^4 & = & x^4+12x^3+54x^2+108x+81 \\ -2y^3=-2(x+3) & = & -2x^3-18x^2-54x-54 \\ y^2=(x+3)^2 & = & x^2+6x+9 \\ -6 & = & -6 \\ \hline y^4-2y^3+y^2-6 & = & x^4+10x^3+37x^2+60x+30, \textit{ Ans.} \end{array}$$

4.
$$\begin{aligned}
x^2&=(s+r)^2 &&=s^2+2sr+r^2\\
ax&=(s+r)a &&= as+ar\\
b&= &&b\\
\hline
x^2+ax+b &&&=r^2+2sr+ar+s^2+as+b\\
&&&=r^2+(2s+a)r+s^2+as+b, \textit{ Ans.}
\end{aligned}$$

5.
$$\begin{aligned}
a^4 &&&=a^4\\
a^3b&=a^3\times a &&=a^4\\
a^2b^2&=a^2\times a^2&&=a^4\\
ab^3&=a\times a^3&&=a^4\\
b^4 &&&=a^4\\
\hline
a^4+a^3b+a^2b^2+ab^3+b^4&=5a^4, \textit{ Ans.}
\end{aligned}$$

6.
$$\begin{aligned}
x^3&=(m+1)^3 &&=m^3+3m^2+3m+1\\
ax^2&=(m-1)(m+1)^2&&=m^3+m^2-m-1\\
a^2x&=(m-1)^2(m+1)&&=m^3-m^2-m+1\\
a^3&=(m-1)^3&&=m^3-3m^2+3m-1\\
\hline
x^3+ax^2+a^2x+a^3 &&&=4m^3+4m=4m(m^2+1), \textit{ Ans.}
\end{aligned}$$

7.
$$\begin{aligned}
x^4&=(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4\\
y^4&=(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4\\
\hline
x^4+y^4&=2a^4+12a^2b^2+2b^4\\
&=2(a^4+6a^2b^2+b^4), \textit{ Ans.}
\end{aligned}$$

8. When $a+b+c=s$, $x+a+b+c=x+s$, and $x-a-b-c=x-(a+b+c)=x-s$.

$$\begin{aligned}
(x+a+b+c)^5&=(x+s)^5=x^5+5x^4s+10x^3s^2+10x^2s^3+5xs^4+s^5\\
(x-a-b-c)^5&=(x-s)^5=x^5-5x^4s+10x^3s^2-10x^2s^3+5xs^4-s^5\\
\hline
(x+a+b+c)^5+(x-a-b-c)^5&=2x^5+20x^3s^2+10xs^4\\
&=2(x^5+10x^3s^2+5xs^4), \textit{ Ans.}
\end{aligned}$$

9.
$$\begin{aligned}
x^3&=(y-2)^3&&=y^3-6y^2+12y-8\\
-7x&=-7(y-2)&&=-7y+14\\
6&&&=6\\
\hline
x^3-7x+6&&&=y^3-6y^2+5y, \textit{ Ans.}
\end{aligned}$$

10.
$$\begin{aligned}
x^5 &= (y+1)^5 = y^5+5y^4+10y^3+10y^2+5y+1\\
-2x^4 &= -2(y+1)^4 = \quad -2y^4-8y^3-12y^2-8y+2\\
3x^3 &= 3(y+1)^3 = \quad 3y^3+9y^2+9y+3\\
-7x^2 &= -7(y+1)^2 = \quad -7x^2-14y-7\\
8x &= 8(y+1) = \quad 8y+8\\
-3 &= \quad -3\\
\hline
x^5-2x^4+3x^3-7x^2+8x-3 &= y^5+3y^4+5y^3, \textit{ Ans.}
\end{aligned}$$

11. Expand before substituting, and we have

$$2(a-b)^2(b-c)^2=2a^2b^2-4ab^3+2b^4-4a^2bc+8ab^2c-4b^3c+2a^2c^2-$$
$$[4abc^2+2b^2c^2$$
$$2(a-b)^2(c-a)^2=2a^2c^2-4abc^2+2b^2c^2-4a^3c+8a^2bc-4ab^2c+2a^4-$$
$$[4a^3b+2a^2b^2$$
$$2(b-c)^2(c-a)^2=2b^2c^2-4bc^3+2c^4-4ab^2c+8abc^2-4ac^3+2a^2b^2-$$
$$[4a^2bc+2a^2c^2$$

If we add the quantities thus obtained, all the terms containing *three* letters each will disappear. Arranging the terms containing a and b according to their powers; also the terms containing b and c according to their powers; also the terms containing a and c according to their powers,—observing to separate the terms $2a^4$, $2b^4$, and $2c^4$, into parts, we have three polynomials, as follows:

$$\begin{aligned}
a^4-4a^3b+6a^2b^2-4ab^3+b^4 &= (a-b)^4=x^4;\\
b^4-4b^3c+6b^2c^2-4bc^3+c^4 &= (b-c)^4=y^4;\\
c^4-4c^3a+6c^2a^2-4ca^3+a^4 &= (c-a)^4=z^4;\\
\text{Sum} \qquad\qquad &= x^4+y^4+z^4, \textit{ Ans.}
\end{aligned}$$

GREATEST COMMON DIVISOR.

(**100**, page 53.)

2.
$$\begin{aligned}
2a^2bc^3 &= 2a^2bc^3\\
6ab^2c^5 &= 3\times 2ab^2c^5\\
10a^3bc^3 &= 5\times 2a^3bc^3
\end{aligned}$$

Hence, $2abc^3$, *Ans.*

3.
$$\begin{aligned}
5x^2y^2z^2 &= 5x^2y^2z^2\\
6x^3yz^2 &= 3\times 2x^3yz^2\\
12x^2yz^3 &= 2\times 2\times 3x^2yz^3
\end{aligned}$$

Hence, x^2yz^2, *Ans.*

4. $$x^2-y^2=(x+y)(x-y)$$
$$x^2-2xy+y^2=\qquad(x-y)(x-y)$$

Hence, $x-y$, *Ans.*

5. $$a^2m-b^2m=\qquad m(a-b)(a+b)$$
$$2ac^2m-2bc^2m=2c^2\times m(a-b)$$

Hence, $m(a-b)$, *Ans.*

6. $$a^2x^3-3a^2x^2+a^2x=\quad a^2x(x^2-3x+1)$$
$$3axz^2-ax^2z^2-az^2=-az^2(x^2-3x+1)$$

Hence, $a(x^2-3x\times1)$, *Ans.*

7. $$16x^2-1=(4x+1)(4x-1)\qquad\text{(by **70**, III.)}$$
$$1-8x+16x^2=\qquad(4x-1)(4x-1)\qquad\text{" " II.}$$

Hence, $4x-1$, *Ans.*

(**105**, page 60.)

FIRST OPERATION.

1.
$$\begin{array}{l|l}
x^4-2x^3-\ 4x^2+\ 11x-\ 6 & x^3-8x^2+17x-10 \\
\underline{x^4-8x^3+17x^2-\ 10x} & x+6 \\
\quad 6x^3-21x^2+\ 21x-\ 6 & \\
\quad \underline{6x^3-48x^2+102x-60} & \\
\qquad 27x^2-\ 81x+54, \text{ 1st remainder.} &
\end{array}$$

Dividing this remainder by 27, we have x^2-3x+2 for the next divisor.

SECOND OPERATION.

$$\begin{array}{l|l}
x^3-8x^2+17x-10 & x^2-3x+2 \\
\underline{x^3-3x^2+\ 2x} & x-5 \\
\quad -5x^2+15x-10 & \\
\quad \underline{-5x^2+15x-10} &
\end{array}$$

Whence, x^2-3x+2, *Ans.*

2. No preparation is necessary for the first operation, and as x is involved to the same power in both of the polynominals, it is immaterial which is taken as a divisor.

1st.

$$\begin{array}{l|l}
6x^3+\ \ x^2-44x+21 & 6x^3-26x^2+46x-42 \\
\underline{6x^3-26x^2+46x-42} & 1 \\
\quad 27x^2-90x+63 &
\end{array}$$

Dividing this remainder by 9, we have $3x^2-10x+7$ for the next divisor.

2d.

$$\begin{array}{l|l}
6x^3-26x^2+46x-42 & 3x^2-10x+7 \\ \cline{2-2}
6x^3-20x^2+14x & 2x-2 \\ \hline
\quad -6x^2+32x-42 & \\
\quad -6x^2+20x-14 & \\ \cline{1-1}
\qquad\qquad 12x-28 &
\end{array}$$

Dividing this last remainder by 4, we have $3x-7$ for the next divisor.

3d.

$$\begin{array}{l|l}
3x^2-10x+7 & 3x-7 \\ \cline{2-2}
3x^2-\ 7x & x-1 \\ \hline
\quad -3x+7 & \\
\quad -3x+7x & \\ \cline{1-1}
\end{array}$$

Hence, $3x-7$, *Ans.*

3. The greater polynomial must be multiplied by $3a$, to render its first term divisible by the first term of the less polynomial.

1st.

$$\begin{array}{l|l}
3ax^3-18a^2x^2+30a^3x-9a^4 & 3ax^2-14a^2x+15a^3 \\ \cline{2-2}
3ax^3-14a^2x^2+15a^3x & x, \quad -4a \\ \hline
\quad -4a^2x^2+15a^3x-9a^4 & \\
\quad -12a^2x^2+45a^3x-27a^4, & \text{new prepared dividend.} \\
\quad -12a^2x^2+56a^3x-60a^4 & \\ \cline{1-1}
\qquad\quad -11a^3x+33a^4 & \\
\qquad\qquad x-3a, & \text{next divisor.}
\end{array}$$

In the above operation the first remainder is multiplied by 3, to render its first term divisible by the first term of the divisor. The last remainder is divided by $-11a^3$ for the next divisor.

2d.

$$\begin{array}{l|l}
3ax^2-14a^2x+15a^3 & x-3a \\ \cline{2-2}
3ax^2-\ 9a^2x & 3ax-5a^2 \\ \hline
\quad -5a^2x+15a^3 & \\
\quad -5a^2x+15a^3 & \\ \cline{1-1}
\end{array}$$

Hence, $x-3a$, *Ans.*

4. 1st.

$$\begin{array}{l|l} x^4-8x^3+14x^2+16x-40 & x^3-8x^2+19x-14 \\ x^4-8x^3+19x^2-14x & x \\ \hline \quad -5x^2+30x-40 & \end{array}$$

Dividing by -5, we have x^2-6x+8 for the second divisor.

2d.

$$\begin{array}{l|l} x^3-8x^2+19x-14 & x^2-6x+8 \\ x^3-6x^2+8x & x-2 \\ \hline -2x^2+11x-14 & \\ -2x^2+12x-16 & \\ \hline \quad -x+2 & \end{array}$$

3d.

$$\begin{array}{l|l} x^2-6x+8 & -x+2 \\ x^2-2x & -x+4 \\ \hline -4x+8 & \\ -4x+8 & \\ \hline \end{array}$$

Hence, $-x+2$, or $x-2$, *Ans.*

1st.

5. $$\begin{array}{l|l} a^3+5a^2+5a+1 & a^3+1 \\ a^3 \qquad\qquad +1 & 1 \\ \hline 5a^2+5a & \end{array}$$

Suppress $(5a)$, and we have $a+1$ for the next divisor.

2d.

$$\begin{array}{l|l} a^3+1 & a+1 \\ a^3+a^2 & a^2-a+1 \\ \hline -a^2+1 & \\ -a^2-a & \\ \hline \quad a+1 & \\ \quad a+1 & \\ \hline \end{array}$$

Hence, $a+1$, *Ans.*

6. Multiply the first polynomial by 2.

1st.

$$\begin{array}{l|l} 4a^4-10a^3b-6a^2b^2+14ab^3+6b^4 & 4a^3-2a^2b-4ab^2-3b^3 \\ 4a^4-\ 2a^3b-4a^2b^2-\ 3ab^3 & a-2b \\ \hline -\ 8a^3b-2a^2b^2+17ab^3+6b^4 & \\ -\ 8a^3b+4a^2b^2+\ 8ab^3+6b^4 & \\ \hline \qquad -6a^2b^2+9ab^3 & \end{array}$$

Suppress $-3ab^2$, and we have, $2a-3b$ for the next divisor.

2d.

$$\begin{array}{l|l} 4a^3-2a^2b-4ab^2-3b^3 & 2a-3b \\ 4a^3-6a^2b & 2a^2+2ab+b^2 \\ \hline 4a^2b-4ab^2 & \\ 4a^2b-6ab^2 & \\ \hline \quad 2ab^2-3b^3 & \\ \quad 2ab^2-3b^3 & \\ \hline \end{array}$$

Hence, $2a-3b$, *Ans.*

7. Multiply the greater polynomial by 4, to render its first term divisible by the first term of the other polynomial.

1st.

$$\begin{array}{l|l}
12x^3-16x^2y+12xy^2-8y^3 & 4x^2-7xy+3y^2 \\
\underline{12x^3-21x^2y+\ \ 9xy^2} & \overline{3x, \quad +5y} \\
\text{Multiply by 4.} \quad 5x^2y+\ \ 3xy^2-\ \ 8y^3 & \\
20x^2y+12xy^2-32y^3 \quad \text{New prepared dividend.} & \\
\underline{20x^2y-35xy^2+15y^3} & \\
\qquad 47xy^2-47y^3 &
\end{array}$$

Dividing by $47y^2$, we have $x-y$ for a new divisor.

2d.

$$\begin{array}{l|l}
4x^2-7xy+3y^2 & x-y \\
\underline{4x^2-4xy} & \overline{4x-3y} \\
\quad -3xy+3y^2 & \\
\quad \underline{-3xy+3y^2} &
\end{array}$$

Hence, $x-y$, *Ans.*

8.
$$\begin{array}{l|l}
4x^5-\ \ 2x^4+\ \ 4x^3-27x^2+\ \ 4x-7 & 2x^4+6x^3-19x^2+4x-5 \\
\underline{4x^5+12x^4-38x^3+\ \ 8x^2-10x} & \overline{2x-7} \\
\quad -14x^4+42x^3-\ \ 35x^2+14x-7 & \\
\quad \underline{-14x^4-42x^3+133x^2-28x+35} & \\
\qquad\qquad 84x^3-168x^2+42x-42. &
\end{array}$$

Dividing the remainder by 42, we have $2x^3-4x^2+x-1$ for the next divisor.

2d.

$$\begin{array}{l|l}
2x^4+6x^3-19x^2+4x-5 & 2x^3-4x^2+x-1 \\
\underline{2x^4-4x^3+\ \ \ x^2-x} & \overline{x+5} \\
\quad 10x^3-20x^2+5x-5 & \\
\quad \underline{10x^3-20x^2+5x-5} &
\end{array}$$

Hence, $2x^3-4x^2+x-1$, *Ans.*

9. The first polynomial contains the monomial factor ac, and the second contains the monomial factor c^2. Suppress these factors, and set aside c, which is common to both, as one factor of the greatest common divisor; then apply the process of division to the resulting polynomials.

1st.

$$\begin{array}{l|l} a^4-4a^2m+3m^2 & a^4-6a^2m+5m^2 \\ a^4-6a^2m+5m^2 & 1 \\ \hline 2a^2m-2m^2 & \end{array}$$

Dividing the remainder by $2m$, we have a^2-m for the next divisor.

2d.

$$\begin{array}{l|l} a^4-6a^2m+5m^2 & a^2-m \\ a^4-a^2m & a^2-5m \\ \hline -5a^2m+5m^2 & \\ -5a^2m+5m^2 & \\ \hline \end{array}$$

Hence, $c(a^2-m)$, *Ans.*

10. Multiply the greater polynomial by 2, to render its first term divisible by the first term of the other polynomial.

1st.

$$\begin{array}{l|l} 2x^4-8x^3-32x^2+14x+48 & 2x^3-15x^2+9x+40 \\ 2x^4-15x^3+9x^2+40x & x,\ +7 \\ \hline 7x^3-41x^2-26x+48 & \\ 14x^3-82x^2-52x+96 & \text{New prepared dividend.} \\ 14x^3-105x^2+63x+280 & \\ \hline 23x^2-115x-184 & \end{array}$$

Divide by 23, and we have x^2-5x-8 for the next divisor.

2d.

$$\begin{array}{l|l} 2x^3-15x^2+9x+40 & x^2-5x-8 \\ 2x^3-10x^2-16x & 2x-5 \\ \hline -5x^2+25x+40 & \\ -5x^2+25x+40 & \end{array}$$

Hence, x^2-5x-8, *Ans.*

1st.

11. $$\begin{array}{l|l} 15x^6+71x^4+60x^2-56 & 3x^6-17x^4-20x^2+84 \\ 15x^6-85x^4-100x^2+420 & 5 \\ \hline \end{array}$$

Suppress 4, $156x^4+160x^2-476$

and we have $39x^4+40x^2-119$ for the next divisor.

Multiply the first divisor by 13 to render division possible, and proceed as follows:

2d.

$$\begin{array}{r|l} 39x^6-221x^4-\ \ 260x^2+1092 & 39x^4+40x^2-119 \\ 39x^6+\ \ 40x^4-\ \ 119x^2\qquad\quad & x^2, +29 \\ \hline \end{array}$$

Suppress -3, $\qquad -261x^4-\ \ 141x^2+1092$

and we have $\qquad 87x^4+\ \ 47x^2-\ \ 364$

Multiplying by 13, we have $1131x^4+\ \ 611x^2-4732$ New prepared dividend.

$$\begin{array}{r} 1131x^4+1160x^2-3451 \\ \hline -549x^2-1281 \end{array}$$

Suppress the factor -183, and we have $3x^2+7$ for the next divisor.

3d.

$$\begin{array}{r|l} 39x^4+40x^2-119 & 3x^2+7 \\ \cline{2-2} 39x^4+91x^2\qquad\quad & 13x^2-17 \\ \hline -51x^2-119 & \\ -51x^2-119 & \end{array}$$

Whence, $3x^2+7$, *Ans.*

12. We will first find the greatest common divisor of the first two polynomials, using the second as dividend, and the first as divisor.

1st.

$$\begin{array}{r|l} 6a^4-14a^2m^2+\ \ 4m^4 & 3a^4+14a^2m^2-5m^4 \\ \cline{2-2} 6a^4+28a^2m^2-10m^4 & 2 \\ \hline -42a^2m^2+14m^4 & \end{array}$$

Suppressing $-14m^2$, we have $3a^2-m^2$ for the next divisor.

2d.

$$\begin{array}{r|l} 3a^4+14a^2m^2-5m^4 & 3a^2-\ \ m^2 \\ \cline{2-2} 3a^4-\ \ \ \ a^2m^2\qquad\quad & a^2+5m^2 \\ \hline +15a^2m^2-5m^4 & \\ 15a^2m^2-5m^4 & \\ \hline \end{array}$$

Hence, $3a^2-m^2$ is the greatest common divisor of the first two polynomials. We must now find the greatest common divisor of this result and the third polynomial, which will be the greatest common divisor required.

3d.

$$
\begin{array}{l|l}
3a^4-22a^2m^2+7m^4 & 3a^2-\ m^2 \\
3a^4-\ \ a^2m^2 & a^2-7m^2 \\
\hline
\quad -21a^2m^2+7m^4 & \\
\quad -21a^2m^2+7m^4 & \\
\hline
\end{array}
$$

Whence, $3a^2-m^2$, *Ans.*

13. The second polynominal contains the monominal factor by. Suppressing this factor, we use the resulting polynominal as the first divisor.

1st.

$$
\begin{array}{l|l}
2a^3x^3-2a^2bx^2y+\ ab^2xy^2-\ b^3y^3 & a^2x^2-2abxy+b^2y^2 \\
2a^3x^3-4a^2bx^2y+2ab^2xy^2 & 2ax+2by \\
\hline
\qquad 2a^2bx^2y-\ ab^2xy^2-\ b^3y^3 & \\
\qquad 2a^2bx^2y-4ab^2xy^2+2b^3y^3 & \\
\hline
\qquad\qquad 3ab^2xy^2-3b^3y^3 & \\
\end{array}
$$

Suppressing $3b^2y^2$ we have $ax-by$ for the next divisor.

2d.

$$
\begin{array}{l|l}
a^2x^2-2abxy+b^2y^2 & ax-by \\
a^2x^2-\ abxy & ax-by \\
\hline
\quad -\ abxy+b^2y^2 & \\
\quad -\ abxy+b^2y^2 & \\
\hline
\end{array}
$$

Whence, $ax-by$. *Ans.*

1st.

14.
$$
\begin{array}{l|l}
9a^4+12a^3+10a^2+\ 4a+1 & 3a^4+8a^3+14a^2+8a+3 \\
9a^4+24a^3+42a^2+24a+9 & 3 \\
\hline
\end{array}
$$

Suppressing -4, $\quad -12a^3-32a^2-20a-8$

we have $\quad 3a^3+\ 8a^2+\ 5a+2$ for the next divisor.

2d.

$$
\begin{array}{l|l}
3a^4+8a^3+14a^2+8a+3 & 3a^3+8a^2+5a+2 \\
3a^4+8a^3+\ 5a^2+2a & a \\
\hline
\end{array}
$$

Suppressing 3, $\quad 9a^2+6a+3$

and we have $\quad 3a^2+2a+1$ for the next divisor.

3d.

$$\begin{array}{l|l} 3a^3+8a^2+5a+2 & 3a^2+2a+1 \\ \cline{2-2} 3a^3+2a^2+\ a & a+2 \\ \cline{1-1} \quad 6a^2+4a+2 & \\ \quad \underline{6a^2+4a+2} & \end{array}$$

Whence, $3a^2+2a+1$, *Ans.*

LEAST COMMON MULTIPLE.

(**109**, Page 62)

2.
$$\begin{aligned} 2a^4bc &= 2a^4bc \\ 5a^2c^2 &= 5a^2c^2 \\ 10ab^3d &= 2\times 5ab^3d \\ 15abcd &= 3\times 5abcd \end{aligned}$$

Hence, $2\times 3\times 5\times a^4b^3c^2d=30a^4b^3c^2d$, *Ans.*

3.
$$\begin{aligned} 3x^2y &= 3x^2y \\ 15xy^2 &= 3\times 5\times xy^2 \\ 10xyz^2 &= 2\times 5\times xyz^2 \\ 5x^2y^2z &= 5x^2y^2z \end{aligned}$$

Hence, $2\times 3\times 5x^2y^2z^2=30x^2y^2z^2$, *Ans.*

4.
$$\begin{aligned} x^2+xy &= x(x+y) \\ xy-y^2 &= \qquad (x-y)y \\ x^2-y^2 &= (x+y)(x-y) \end{aligned}$$

Hence, $(x+y)(x-y)xy=x^3y-xy^3$, *Ans.*

5.
$$\begin{aligned} x^4-a^4 &= (x^2+a^2)(x+a)(x-a) \\ x^2-a^2 &= \qquad\qquad (x+a)(x-a) \\ x^2+a^2 &= (x^2+a^2) \\ x^4-2a^2x^2+a^4 &= \qquad\qquad (x+a)^2(x-a)^2 \end{aligned}$$

Hence, $(x^2+a^2)(x+a)^2(x-a)^2=x^6-a^2x^4-a^4x^2+a^6$, *Ans.*

6.
$$\begin{aligned} x^3-x &= x(x+1)(x-1) \\ x^3-1 &= (x^2+x+1)(x-1) \\ x^3+1 &= (x^2-x+1)(x+1) \end{aligned}$$

Hence, $x(x^2+x+1)(x^2-x+1)(x+1)(x-1)=x(x^6-1)=x^7-x$, *Ans.*

7.
$$x^4+2x^2+1=(x^2+1)^2$$
$$x^4-2x^2+1=(x^2-1)^2=(x+1)^2(x-1)^2$$
$$x^2+2x+1=\qquad(x+1)^2$$
$$x^2-2x+1=\qquad(x-1)^2$$
$$x+1=\qquad(x+1)$$
$$x-1=\qquad(x-1)$$

Hence, $(x^2+1)^2(x+1)^2(x-1)^2=x^8-2x^4+1$, *Ans.*

8.
$$4x^3+2x=2x(2x^2+1)$$
$$6x^2-4x=2x(3x-2)$$
$$6x^2+4x=2x(3x+2)$$

Hence, $2x(2x^2+1)(3x-2)(3x+2)=36x^5+2x^3-8x$, *Ans.*

9.
$$x^2-4a^2=(x+2a)(x-2a)$$
$$(x+2a)^3=(x+2a)^3$$
$$(x-2a)^3=\qquad(x-2a)^3$$

Hence, $(x+2a)^3(x-2a)^3=(x^2-4a^2)^3=x^6-12x^4a^2+48x^2a^4-64a^6$, *Ans.*

10.
$$a^4-b^4=(a^2+b^2)(a+b)(a-b)$$
$$a^3-b^3=\qquad(a-b)(a^2+ab+b^2)$$
$$a^2-b^2=\qquad(a+b)(a-b)$$
$$a-b=\qquad(a-b)$$

Hence, $(a^2+b^2)(a+b)(a-b)(a^2+ab+b^2)=a^6+a^5b+a^4b^2-a^2b^4-ab^5-b^6$, *Ans.*

(**110**, page 63.)

1. The greatest common divisor of the two polynomials, (**105**), is x^2-2x+2;

$$(x^3-5x^2+8x-6)\div(x^2-2x+2)=x-3;\text{ hence,}$$
$$(x-3)(x^3+x^2-4x+6)=x^4-2x^3-7x^2+18x-18,\ \textit{Ans.}$$

2. The greatest common divisor of the given polynomials, (**105**), is $x-5$;

$$(x^2-12x+35)\div(x-5)=x-7;\text{ hence,}$$
$$(x-7)(x^3-2x^2-19x+20)=x^4-9x^3-5x^2+153x-140,\ \textit{Ans.}$$

3. The greatest common divisor of the given polynomials, (**105**), is $2am^2-1$;

$(2a^2m^4+3am^2-2)\div(2am^2-1)=am^2+2$; hence,

$$(am^2+2)(6a^2m^4-am^2-1)=6a^3m^6+11a^2m^4-3am^2-2, \textit{ Ans.}$$

4. The greatest common divisor of the given polynomials is x^2-3x+1;

$$(x^3-5x^2+7x-2)\div(x^2-3x+1)=x-2; \text{ hence,}$$
$$(x-2)(2x^3-5x^2-x+1)=2x^4-9x^3+9x^2+3x-2, \textit{ Ans.}$$

5. The greatest common divisor of the given polynomials is $3x^2-5$;

$$(3x^3+6x^2-5x-10)\div(3x^2-5)=x+2; \text{ hence,}$$
$$(x+2)(6x^4-4x^2-10)=6x^5+12x^4-4x^3-8x^2-10x-20, \textit{ Ans.}$$

6. The greatest common divisor of first two polynomials is $x+2$;

$$(x^2-2x-8)\div(x+2)=(x-4);$$

hence, $(x-4)(x^2+7x+10)=x^3+3x^2-18x-40$, the least common multiple of the first two polynomials. We now proceed to find the least common multiple of this result, and the third polynomial.

The greatest common divisor, &c., is x^2+x-20;

$$(x^2+x-20)\div(x^2+x-20)=1; \text{ hence,}$$
$$(x^3+3x^2-18x-40)\times 1=x^3+3x^2-18x-40, \textit{ Ans.}$$

7. The greatest common divisor of the first two polynomials is $a-2b$; and

$$(a^2-ab-2b^2)\div(a-2b)=a+b; \text{ hence,}$$
$$(a+b)(a^2-3ab+2b^2)=a^3-2a^2b-ab^2+2b^3,$$

which is the least common multiple of the first two polynomials.

The greatest common divisor of this result and the third polynomial is a^2-b^2; hence,

$$a^3-2a^2b-ab^2+2b^3, \textit{ Ans.}$$

8. The greatest common divisor of the first two polynomials is $2x-y$, and $(2x^2-5xy+2y^2)\div(2x-y)=x-2y$. Hence, we have

$$(x-2y)(2x^2-7xy+3y^2)=2x^3-11x^2y+17xy^2-6y^3,$$

which is the least common multiple of the first two polynomials.

The greatest common divisor of this result and the third polynomial is $x^2-5xy+6y^2$; hence,

$$2x^3-11x^2y+17xy^2-6y^3, \textit{ Ans.}$$

FRACTIONS.

REDUCTION.

(**124**, page 67.)

3. $\dfrac{7x^3yz}{21xy^3z}=\dfrac{(7xyz)x^2}{(7xyz)3y^2}=\dfrac{x^2}{3y^2}$, *Ans.*

4. $\dfrac{x^2-1}{xy+y}=\dfrac{(x+1)(x-1)}{(x+1)y}=\dfrac{x-1}{y}$, *Ans.*

5. $\dfrac{a^3-ab^2}{a^2+2ab+b^2}=\dfrac{a(a-b)(a+b)}{(a+b)(a+b)}=\dfrac{a(a-b)}{a+b}=\dfrac{a^2-ab}{a+b}$, *Ans.*

6. $\dfrac{x^5-b^2x^3}{x^4-b^4}=\dfrac{x^3(x^2-b^2)}{(x^2+b^2)(x^2-b^2)}=\dfrac{x^3}{x^2+b^2}$, *Ans.*

7. $\dfrac{2x^2-16x-6}{3x^2-24x-9}=\dfrac{2(x^2-8x-3)}{3(x^2-8x-3)}=\dfrac{2}{3}$, *Ans.*

8. The greatest common divisor of the numerator and denominator, found by (**105**), is $2x-3$; hence,

$$(2x^3-7x^2+14x-12)\div(2x-3)=x^2-2x+4$$
$$(4x^3-4x^2-13x+15)\div(2x-3)=2x^2+x-5$$

And we have for the reduced fraction, $\dfrac{x^2-2x+4}{2x^2+x-5}$, *Ans.*

9. $\dfrac{a^2c+2abc+b^2c}{a^3+3a^2b+3ab^2+b^3}=\dfrac{c(a+b)(a+b)}{(a+b)(a+b)(a+b)}=\dfrac{c}{a+b}$, *Ans.*

10. $\dfrac{a^3-3a^2x+3ax^2-x^3}{a^2-x^2}=\dfrac{(a-x)^3}{a^2-x^2}=\dfrac{(a-x)(a-x)^2}{(a-x)(a+x)}=\dfrac{a^2-2ax+x^2}{a+x}$, *Ans.*

11. The greatest common divisor of the numerator and denominator is $2a^2+3x$; hence,

$$(6a^2+\ \ 7ax-3x^2)\div(2a+3x)=3a-x$$
$$(6a^2+11ax+3x^2)\div(2a+3x)=3a+x$$

And we have for the reduced fraction, $\dfrac{3a-x}{3a+x}$, *Ans.*

12. The greatest common divisor of the numerator and denominator is x^3-x^2-x+1; hence,

$$(x^5-x^4-x+1)\div(x^3-x^2-x+1)=x^2+1$$
$$(x^4-x^3-x^2+x)\div(x^3-x^2-x+1)=x$$

And we have for the reduced fraction, $\dfrac{x^2+1}{x}$, *Ans.*

13. $$\frac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}=\frac{(x+y)^5-(x^5+y^5)}{(x+y)^3-(x^3+y^3)}$$
$$=\frac{(x+y)[(x+y)^4-(x^4-x^3y+x^2y^2-xy^3+y^4)]}{(x+y)[(x+y)^2-(x^2-xy+y^2)]}$$
$$=\frac{5x^3y+5x^2y^2+5xy^3}{3xy}$$
$$=\frac{5(x^2+xy+y^2)}{3}, \textit{ Ans.}$$

14. Removing the parentheses by multiplication and involution, we have

$$\frac{(3x^2-1)(2x^2-1)-x^2(5x^2-7)}{(3x^2-1)^2+(x^3-3x)^2}=\frac{6x^4-5x^2+1-5x^4+7x^2}{9x^4-6x^2+1+x^6-6x^4+9x^2}=$$
$$\frac{x^4+2x^2+1}{x^6+3x^4+3x^2+1}=\frac{(x^2+1)^2}{(x^2+1)^3}=\frac{1}{x^2+1}, \textit{ Ans.}$$

(**125**, page 68.)

4. $$\frac{2x^3-2y^3}{x-y}=\frac{2(x-y)(x^2+xy+y^2)}{x-y}=2(x^2+xy+y^2), \textit{ Ans.}$$

6. $$\frac{24x^2-18x-6}{8x}=3x-2-\frac{2x}{8}-\frac{6}{8}$$
$$=3x-2-\frac{2x+6}{8}$$
$$=3x-2-\frac{x+3}{4}, \textit{ Ans.}$$

8. $$\frac{56x^2+126x-140}{7x+21}=8x-6+\frac{-14}{7x+21}$$
$$=8x-6-\frac{2}{x+3}, \textit{ Ans.}$$

10. $\frac{x^7-y^7}{x^3-y^3}=x^4+xy^3+\frac{xy^6-y^7}{x^3-y^3}$

$$=x^4+xy^3+\frac{y^6(x-y)}{(x^2+xy+y^2)(x-y)}$$

$$=x^4+xy^3+\frac{y^6}{x^2+xy+y^2}, \text{ Ans.}$$

(**126**, page 69.)

1. $1+a+\frac{a^2}{b}=\frac{b+ab+a^2}{b}$, *Ans.*

2. $2b-\frac{3x-a}{b}=\frac{2bc-3x+a}{c}$, *Ans.*

3. $5a+\frac{ab+x}{b}=\frac{5ab+ab+x}{b}=\frac{6ab+x}{b}$, *Ans.*

4. $12+\frac{3a+b}{b}=\frac{12b+3a+b}{b}=\frac{13b+3a}{b}$, *Ans.*

5. $5x-\frac{2x-5}{3}=\frac{15x-(2x-5)}{3}=\frac{13x+5}{3}$, *Ans.*

6. $3a-9-\frac{3a^2-30}{a+3}=\frac{3a^2-27-(3a^2-30)}{a+3}=\frac{3}{a+3}$. *Ans.*

7. $x+y+\frac{y^2}{x-y}=\frac{x^2-y^2+y^2}{x-y}=\frac{x^2}{x-y}$, *Ans.*

8. $x+1-\frac{x^3-4x^2+8}{(x-2)^2}=\frac{x^3-3x^2+4-(x^3-4x^2+8)}{(x-2)^2}=\frac{x^2-4}{(x-2)^2}=\frac{x+2}{x-2}$, *Ans*

9. $a^2+ab+b^2-\frac{a^3+b^3}{a-b}=\frac{a^3-b^3-(a^3+b^3)}{a-b}=-\frac{2b^3}{a-b}$, *Ans.*

10. $1+2y+2y^2+2y^3+\frac{2y^4+2y^5}{1-y^2}=\frac{1+2y+2y^2}{1-y^2}=\frac{(1+y)(1+y)}{(1+y)(1-y)}=\frac{1+y}{1-y}$, *Ans.*

11. $(x-1)^2-\dfrac{(x-1)^2}{x}=\dfrac{x(x-1)^2-(x-1)^2}{x}=\dfrac{(x-1)(x-1)^2}{x}$.

$$=\frac{(x-1)^4}{x}, \textit{ Ans.}$$

12. $x^2+5xy+y^2+\dfrac{21x^2y^2}{x^2-5xy+y^2}=\dfrac{x^4+2x^2y^2-25x^2y^2+y^4+21x^2y^2}{x^2-5xy+y^2}=$

$$\frac{x^4-2x^2y^2+y^4}{x^2-5xy+y^2}=\frac{(x^2-y^2)^2}{x^2-5xy+y^2}, \textit{ Ans.}$$

ADDITION.

(129, page 75.)

1. $\dfrac{3x}{5}+\dfrac{2x}{7}+\dfrac{x}{3}=\dfrac{63x+30x+35x}{105}=\dfrac{128x}{105}$, *Ans.*

2. $\dfrac{a}{b}+\dfrac{a+b}{c}=\dfrac{ac+ab+b^2}{bc}$, *Ans.*

3. $\dfrac{a^2}{3}+\dfrac{a^2+x^2}{a+x}=\dfrac{a^3+a^2x+3a^2+3x^2}{3(a+x)}$, *Ans.*

4. $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}=\dfrac{a^2+2ab+b^2+a^2-2ab+b^2}{a^2-b^2}=\dfrac{2a^2+2b^2}{a^2-b^2}$, *Ans.*

5. The sum of the entire quantities is $6a$.

$$\frac{a+3}{5}+\frac{2a-5}{4}=\frac{4a+12+10a-25}{20}=\frac{14a-13}{20}.$$

Hence, $\qquad 6a+\dfrac{14a-13}{20}$, *Ans.*

6. $$\frac{x-2}{3}+\frac{2x-3}{5x}=\frac{5x^2-10x+6x-9}{15x}=\frac{5x^2-4x-9}{15x}.$$

Hence, $\qquad 9x+\dfrac{5x^2-4x-9}{15x}$, *Ans.*

7. $\dfrac{a}{a+c}+\dfrac{2c}{a-c}+\dfrac{c}{a+c}=\dfrac{a^2-ac+2ac+2c^2+ac-c^2}{a^2-c^2}$

$$=\frac{a^2+2ac+c^2}{a^2-c^2}=\frac{a+c}{a-c}, \textit{ Ans.}$$

8. $\dfrac{x^2y-3y^2}{5x^2}+\dfrac{3x^4+3y^4}{5x^2y^2}+\dfrac{xy^2-6x^2}{10y^2}=$

$$\frac{2x^2y^3-6y^4+6x^4+6y^4+x^3y^2-6x^4}{10x^2y^2}=\frac{2x^2y^3+x^3y^2}{10x^2y^2}=\frac{2y+x}{10},\ \textit{Ans.}$$

9. We find, by inspection, that the least common multiple of the given denominators is $(b-c)(c-a)(a-b)$. We must, therefore, multiply the numerator and denominator of the first fraction by $(a-b)$, of the second by $(b-c)$, and of the third by $(c-a)$, to reduce the fractions to their least common denominator. Thus,

$$\frac{a+b}{(b-c)(c-a)}=\frac{a^2-b^2}{(b-c)(c-a)(a-b)};$$

$$\frac{b+c}{(c-a)(a-b)}=\frac{b^2-c^2}{(b-c)(c-a)(a-b)};$$

$$\frac{c+a}{(a-b)(b-c)}=\frac{c^2-a^2}{(b-c)(c-a)(a-b)}.$$

Taking the sum of the numerators, we have

$$\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(b-c)(c-a)(a-b)}=\frac{0}{(b-c)(c-a)(a-b)}=0,\ \textit{Ans.}$$

10. By (**109**), we find that the least common multiple of the given denominators is $(a-b)(a-1)(b+1)$. Hence the terms of the first fraction must be multiplied by $b+1$, of the second by $-(a-1)$, and of the third by $-(a-b)$. Thus,

$$\frac{a^2-b}{(a-b)(a-1)}=\frac{a^2b-b^2+a^2-b}{(a-b)(a-1)(b+1)};$$

$$\frac{b^2+a}{(b+1)(b-a)}=\frac{-ab^2-a^2+b^2+a}{(a-b)(a-1)(b+1)};$$

$$\frac{1+ab}{(1-a)(1+b)}=\frac{-a-a^2b+b+ab^2}{(a-b)(a-1)(b+1)}.$$

Taking the sum of the numerators, we have

$$\frac{a^2b-b^2+a^2-b-ab^2-a^2+b^2+a-a-a^2b}{(a-b)(a-1)(b+1)}=\frac{0}{(a-b)(a-1)(b+1)}=0,$$ *Ans.*

11. The least common denominator is $(a-b)(a-c)(b-c)$.

$$\frac{bc}{(a-b)(a-c)}=\frac{b^2c-bc^2}{(a-b)(a-c)(b-c)};$$

$$\frac{ac}{(b-c)(b-a)}=\frac{ac^2-a^2c}{(a-b)(a-c)(b-c)};$$

$$\frac{ab}{(c-a)(c-b)}=\frac{a^2b-ab^2}{(a-b)(a-c)(b-c)}.$$

Their sum is $\dfrac{a^2b-ab^2+b^2c-bc^2+ac^2-a^2c}{a^2b-ab^2+b^2c-bc^2+ac^2-a^2c}=1$, *Ans.*

12. By (**110**), we find the least common multiple of the given denominators to be $x^3-6x^2+11x-6$. Dividing this by the denominators of the given fractions, the respective quotients are $(x-3)$, $(x-2)$, and $(x-1)$. Multiplying the terms of the first fraction by $(x-3)$, of the second by $(x-2)$, and of the third by $(x-1)$, we have

$$\frac{x-3}{x^2-3x+2}=\frac{x^2-6x+9}{x^3-6x^2+11x-6};$$

$$\frac{x-2}{x^2-4x+3}=\frac{x^2-4x+4}{x^3-6x^2+11x-6};$$

$$\frac{x-1}{x^2-5x+6}=\frac{x^2-2x+1}{x^3-6x^2+11x-6}.$$

Taking the sum of the numerators, we have

$$\frac{x^2-6x+9+x^2-4x+4+x^2-2x+1}{x^3-6x^2+11x-6}=\frac{3x^2-12x+14}{x^3-6x^2+11x-6},\ \textit{Ans.}$$

13. The least common multiple of the given denominators is found to be $x^3+6x^2+11x+6$. Hence, multiply the terms of the first fraction by (x^2+5x+6), of the second by $(x+3)$, and of the third by 1.

$$\frac{x}{x+1}=\frac{x^3+5x^2+6x}{x^3+6x^2+11x+6};$$

$$\frac{x^2}{x^2+3x+2}=\frac{x^3+3x^2}{x^3+6x^2+11x+6};$$

$$\frac{x^3-2x^2-3x}{x^3+6x^2+11x+6}=\frac{x^3-2x^2-3x}{x^3+6x^2+11x+6}.$$

Taking the sum of the numerators, we have

$$\frac{x^3+5x^2+6x+x^3+3x^2+x^3-2x^2-3x}{x^3+6x^2+11x+6}=\frac{3x^3+6x^2+3x}{x^3+6x^2+11x+6}=$$

$$\frac{3x(x^2+2x+1)}{(x+1)(x^2+5x+6)}=\frac{3x(x+1)(x+1)}{(x+1)(x^2+5x+6)}=\frac{3x(x+1)}{x^2+5x+6},\ \textit{Ans.}$$

SUBTRACTION.

(**130**, page 76.)

1. $\frac{3x}{7}-\frac{2x}{9}=\frac{27x-14x}{63}=\frac{12x}{63}$, *Ans.*

2. $\frac{7x}{2}-\frac{2x-1}{3}=\frac{21x-(4x-2)}{6}=\frac{17x+2}{6}$, *Ans.*

3. $\frac{1}{x-y}-\frac{1}{x+y}=\frac{x+y-(x-y)}{x^2-y^2}=\frac{2y}{x^2-y^2}$, *Ans.*

4. The difference of the entire quantities is a, and the difference of the fractions is

$$\frac{11a-10}{15}-\frac{3a-5}{7}=\frac{77a-70-(45a-75)}{105}=\frac{32a+5}{105}$$

Hence, $a+\frac{32a+5}{105}$, *Ans.*

5. $\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{a^2+2ab+b^2-(a^2-2ab+b^2)}{a^2-b^2}=\frac{4ab}{a^2-b^2}$, *Ans.*

6. We first take the difference of the fractions.

$$\frac{-y}{+xy}-\frac{x+y}{x^2-xy}=\frac{x^2-2xy+y^2}{x(x^2-y^2)}-\frac{(x^2+2xy+y^2)}{x(x^3-y^2)}=\frac{-4xy}{x(x^2-y^2)}=\frac{-4y}{x^2-y^2}.$$

Hence, $x-\frac{4y}{(x^2-y^2)}$, *Ans.*

7. The difference of the entire quantities is $2x$. The difference of the fractions is

$$\frac{x}{b}-\left(-\frac{x-a}{c}\right)=\frac{x}{b}+\frac{x-a}{c}=\frac{cx+bx-ab}{bc};$$

Hence, $$2y+\frac{cx+bx-ab}{bc}, \textit{ Ans.}$$

8. By (**105**), we find the greatest common divisor of the denominators to be $(2x-3)$.

$(2x^2-11x+12)\div(2x-3)=x-4$; $(2x^2+5x-12)\div(2x-3)=x+4$.

Hence, the least common denominator of the fractions is

$(2x-3)(x-4)(x+4)$; and we have

$$\frac{x^2+x-5}{2x^2-11x+12}=\frac{x^2+x-5}{(2x-3)(x-4)}=\frac{x^3+5x^2-x-20}{(2x-3)(x-4)(x+4)};$$

$$\frac{x^2+x-1}{2x^2+5x-12}=\frac{x^2+x-1}{(2x-3)(x+4)}=\frac{x^3-3x^2-5x+4}{(2x-3)(x-4)(x+4)}.$$

Taking the difference of the numerators, we have

$$\frac{x^3+5x^2-x-20-(x^3-3x^2-5x+4)}{(2x-3)(x-4)(x+4)}=\frac{8x^2+4x-24}{(2x-3)(x-4)(x+4)}$$

$$=\frac{(2x-3)(4x+8)}{(2x-3)(x^2-16)}=\frac{4x+8}{x^2-16}, \textit{ Ans.}$$

9. We may factor the denominators of the given fractions by inspection. Thus,

$$a^2+3ab+2b^2=(a+b)(a+2b);$$
$$a^2+5ab+6b^2=(a+2b)(a+3b).$$

Hence $(a+b)(a+2b)(a+3b)$ is the least common denominator of the fractions; and we have

$$\frac{3a+b}{a^2+3ab+2b^2}=\frac{3a^2+10ab+3b^2}{(a+b)(a+2b)(a+3b)};$$

$$\frac{a+7b}{a^2+5ab+6b^2}=\frac{a^2+8ab+7b^2}{(a+b)(a+2b)(a+3b)}.$$ Hence,

$$\frac{3a^2+10ab+3b^2-(a^2+8ab+7b^2)}{(a+b)(a+2b)(a+3b)}=\frac{2a^2+2ab-4b^2}{(a+b)(a+2b)(a+3b)}$$
$$=\frac{2(a+2b)(a-b)}{(a+b)(a+2b)(a+3b)}$$
$$=\frac{2(a-b)}{a^2+4ab+3b^2},\ \textit{Ans.}$$

10. The denominators may be transformed as follows:

$$7ab(a-b)-2(a^3-b^3)=(a-b)[7ab-2(a^2+ab+b^2)]=(a-b)(5ab-2a^2-2b^2);$$
$$3ab(a+b)-2(a^3+b^3)=(a+b)[3ab-2(a^2-ab+b^2)]=(a+b)(5ab-2a^2-2b^2).$$

Hence the least common denominator is $(a^2-b^2)(5ab-2a^2-2b^2)$; and the terms of the first fraction must be multiplied by $(a+b)$, and the terms of the second fraction by $(a-b)$. We shall have

$$\frac{4a-3b}{7ab(a-b)-2(a^3+b^3)}=\frac{4a^2+ab-3b^2}{(a^2-b^2)(5ab-2a^2-2b^2)};$$
$$\frac{8a-b}{3ab(a+b)-2(a^3+b^3)}=\frac{8a^2-9ab+b^2}{(a^2-b^2)(5ab-2a^2-2b^2)}.$$

And the difference is

$$\frac{4a^2+ab-3b^2-(8a^2-9ab+b^2)}{(a^2-b^2)(5ab-2a^2-2b^2)}=\frac{10ab-4a^2-4b^2}{(a^2-b^2)(5ab-2a^2-2b^2)}=\frac{2}{a^2-b^2},$$
Ans.

MULTIPLICATION.

(132, page 77.)

1. Multiply $\frac{a}{b}$ by $\frac{b}{x}$.

Canceling the common factor b, we have $\frac{a}{x}$, *Ans.*

2. $\frac{a+x}{30}\times\frac{5a}{3(a+x)}=\frac{a+x}{6\times5}\times\frac{5a}{3(a+x)}=\frac{a}{18}$, *Ans.*

3. $\frac{2x+3y}{2a}\times\frac{2a}{5x}=\frac{2x+3y}{5x}$, *Ans.*

4. $\frac{a^2-x^2}{2y}\times\frac{2a}{a+x}=\frac{(a+x)(a-x)}{2y}\times\frac{2a}{a+x}=\frac{(a-x)a}{y}$, *Ans.*

5. $\frac{4y^2}{5y-10}\times\frac{15y-30}{2y}=\frac{2y\times 2y}{5y-10}\times\frac{3(5y-10)}{2y}=6y$, *Ans.*

6. $\frac{a^4-b^4}{a+b}\times\frac{a^2}{ab-b^2}=\frac{(a^2+b^2)(a+b)(a-b)}{(a+b)}\times\frac{a^2}{b(a-b)}=\frac{a^2(a^2+b^2)}{b}$, *Ans.*

7. $\frac{a^2x-x^3}{a}\times\frac{6a}{2ax-2x^2}=\frac{x(a+x)(a-x)}{a}\times\frac{6a}{2x(a-x)}=3(a+x)$, *Ans.*

8. $a+\frac{x}{b}=\frac{ab+x}{b}$, and $a-\frac{y}{b}=\frac{ab-y}{b}$.

$$\frac{ab+x}{b}\times\frac{ab-y}{b}=\frac{a^2b^2+abx-aby-xy}{b^2}, \textit{ Ans.}$$

9. $\frac{3x^2-5x}{14}\times\frac{7a}{2x^3-3x}=\frac{x(3x-5)}{7\times 2}\times\frac{7a}{x(2x^2-3)}=\frac{a(3x-5)}{2(2x^2-3)}$

$$=\frac{3ax-5a}{4x^2-6}, \textit{ Ans.}$$

10. $\frac{x^2-y^2}{x}\times\frac{x}{x+y}\times\frac{a}{x-y}=\frac{(x+y)(x-y)}{x}\times\frac{x}{x+y}\times\frac{a}{x-y}=a$, *Ans.*

11. $\frac{4a^2-16b^2}{a-2b}\times\frac{5a}{8a^2+32ab+32b^2}=\frac{4(a+2b)(a-2b)}{(a-2b)}\times$

$$\frac{5b}{2\times 4(a+2b)(a+2b)}=\frac{5b}{2(a+2b)}=\frac{5b}{2a+4b}, \textit{ Ans.}$$

12. $\frac{x+1}{2a}\times\frac{x-1}{a+b}\times\frac{3a}{1}=\frac{3(x^2-1)}{2(a+b)}$, *Ans.*

13. $a+\frac{ax}{a-x}=\frac{a^2}{a-x}$. Hence, $\frac{a^2-x^2}{a+b}\times\frac{a^2-b^2}{ax+x^2}\times\frac{a^2}{a-x}=$

$$\frac{(a+x)(a-x)}{a+b}\times\frac{(a+b)(a-b)}{x(a+x)}\times\frac{a^2}{a-x}=\frac{a^2(a-b)}{x}, \textit{ Ans.}$$

14. $\dfrac{a^4-x^4}{a^2-b^2}\times\dfrac{a+b}{a^2+x^2}\times\dfrac{a-b}{a-x}=\dfrac{(a^2+x^2)(a+x)(a-x)}{(a+b)(a-b)}\times\dfrac{a+b}{a^2+x^2}\times$

$\dfrac{a-b}{a-x}=a+x$, *Ans.*

15. $\dfrac{x^2-b^2}{bc}\times\dfrac{x^2+b^2}{b+c}\times\dfrac{bc}{x-b}=\dfrac{(x+b)(x-b)}{bc}\times\dfrac{x^2+b^2}{b+c}\times\dfrac{bc}{x-b}=$

$\dfrac{(x+b)(x^2+b^2)}{b+c}$, *Ans.*

16. $\dfrac{c(a-c)}{a^2+2ac+c^2}\times\dfrac{c(a+c)}{a^2-2ac+c^2}\times\dfrac{a^2-c^2}{ac^2x}$

$=\dfrac{c(a-c)}{(a+c)(a+c)}\times\dfrac{c(a+c)}{(a-c)(a-c)}\times\dfrac{(a+c)(a-c)}{accx}=\dfrac{1}{ax}$, *Ans.*

17. $\dfrac{(a+b-c)(a-b+c)}{a-b-c}\times\dfrac{c+b-a}{(c-b-a)(b-c-a)}$

$=\dfrac{(a+b-c)(a-b+c)}{(-1)(c+b-a)}\times\dfrac{c+b-a}{(-1)(a+b-c)\times(-1)(a-b+c)}$

$=\dfrac{(1)(1)(1)}{(-1)(-1)(-1)}=\dfrac{1}{-1}=-1$, *Ans.*

DIVISION.

(**133**, page 80.)

3. $\dfrac{15ab}{a-x}\div\dfrac{10ac}{a^2-x^2}=\dfrac{15ab}{a-x}\times\dfrac{(a+x)(a-x)}{10ac}=\dfrac{3b(a+x)}{2c}$, *Ans.*

4. $\dfrac{2x^2-7}{x+a}\div\dfrac{a^2}{x^2+2ax+a^2}=\dfrac{2x^2-7}{x+a}\times\dfrac{(x+a)(x+a)}{a^2}=$

$\dfrac{(2x^2-7)(x+a)}{a^2}$, *Ans.*

5. $\dfrac{x^4-b^4}{x^2-2bx+b^2}\div\dfrac{x+b}{x-b}=\dfrac{(x^2+b^2)(x+b)(x-b)}{(x-b)(x-b)}\times\dfrac{x-b}{x+b}=x^2+b^2$,

Ans.

6. $\frac{2ax+x^2}{a^3-x^3} \div \frac{x}{a-x} = \frac{x(2a+x)}{(a-x)(a^2+ax+x^2)} \times \frac{a-x}{x} = \frac{2a+x}{a^2+ax+x^2}$, *Ans.*

7. $\frac{14x-3}{5} \div \frac{10x-4}{25} = \frac{14x-3}{5} \times \frac{5\times5}{10x-4} = \frac{70x-15}{10x-4}$, *Ans.*

8. $\frac{9x^2-3x}{5} \div \frac{x^2}{5} = \frac{x(9x-3)}{5} \times \frac{5}{x^2} = \frac{9x-3}{x}$, *Ans.*

9. $\frac{6x-7}{x+1} \div \frac{x-1}{3} = \frac{6x-7}{x+1} \times \frac{3}{x-1} = \frac{18x-21}{x^2-1}$, *Ans.*

10. $\frac{x+x^2}{3a^2} \div \frac{2ax+2ax^2}{7} = \frac{x+x^2}{3a^2} \times \frac{7}{2a(x+x^2)} = \frac{7}{6a^3}$, *Ans.*

11. $\frac{a^6-x^6}{a^2-2ax+x^2} \div \frac{a^2+ax+x^2}{a-x} = \frac{a^3+x^3)(a-x)(a^2+ax+x^2)}{(a-x)(a-x)} \times$
$\frac{a-x}{a^2+ax+x^2} = a^3+x^3$, *Ans.*

12. $\frac{9y^2-3y}{5} \div \frac{y^2}{5} = \frac{y(9y-3)}{5} \times \frac{5}{y \times y} = \frac{9y-3}{y}$, *Ans.*

13. $\frac{na-nx}{a+b} \div \frac{ma-mx}{a+b} = \frac{n(a-x)}{a+b} \times \frac{a+b}{m(a-x)} = \frac{n}{m}$, *Ans.*

14. $a \div \frac{x}{x+y} \times \frac{a}{x-y} = \frac{a}{1} \times \frac{x+y}{x} \times \frac{x-y}{a} = \frac{x^2-y^2}{x}$, *Ans.*

15. $\frac{3(x^2-1)}{2(a+b)} \div \left(\frac{x+1}{2a}\right)\left(\frac{x-1}{a+b}\right) = \frac{3(x+1)(x-1)}{2(a+b)} \times \frac{2a(a+b}{(x+1)(x-1)} =$
$3a$, *Ans.*

16. $\frac{10ab+3a^2+3b^2}{10ab-3a^2-3b^2} \div \left(\frac{3a+b}{b-3a}\right)\left(\frac{b}{a}\right) = \frac{10ab+3a^2+3b^2}{10ab-3a^2-3b^2} \times \frac{a(b-3a)}{b(3a+b)} =$
$\frac{(3a+b)(a+3b)}{(b-3a)(a-3b)} \times \frac{a(b-3a)}{b(3a+b)} = \frac{3ab+a^2}{ab-3b^2}$, *Ans.*

17. $\frac{a^2}{x^3} + \frac{1}{a} = \frac{a^3+x^3}{ax^3}$; $\frac{a}{x^2} - \frac{1}{x} + \frac{1}{a} = \frac{a^2-ax+x^2}{ax^2}$; hence,

$\frac{a^3+x^3}{ax^3} \div \frac{a^2-ax+x^2}{ax^2} = \frac{(a+x)(a^2-ax+x^2}{(ax^2)(x)} \times \frac{ax^2}{a^2-ax+x^2} = \frac{a+x}{x}$, *Ans.*

18. $\frac{a-1}{a}+\frac{b-1}{b}+\frac{c-1}{c}-1=1-\frac{1}{a}+1-\frac{1}{b}+1-\frac{1}{c}-1=2-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$; and $\left[2-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right]\div\left[2-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right]=1$, *Ans.*

REDUCTION OF COMPLEX FORMS.

(**135**, page 81.)

2. Multiplying both numerator and denominator by bc, we have

$$\frac{a+\frac{b}{c}}{a+\frac{c}{b}}=\frac{abc+b^2}{abc+c^2},\ \textit{Ans.}$$

3. Multiplying both numerator and denominator by $a^2b^2c^2$, we have

$$\frac{\frac{a^2}{bc^2}+\frac{b^2}{a^2c}}{\frac{a^2}{b^2c}+\frac{b^2}{ac^2}}=\frac{a^4b+b^4c}{a^4c+b^4a},\ \textit{Ans.}$$

4. Multiplying both numerator and denominator by mn, we have

$$\frac{\frac{x-1}{m}-\frac{x+1}{n}}{\frac{x+1}{m}+\frac{x-1}{n}}=\frac{nx-n-mx-m}{nx+n+mx-m}=\frac{x(n-m)-(n+m)}{x(n+m)+(n-m)},\ \textit{Ans.}$$

5. Multiplying both numerator and denominator by ab, we have

$$\frac{\frac{a+1}{b}-2+\frac{b-1}{a}}{\frac{a-1}{b}-2+\frac{b+1}{a}}=\frac{a^2+a-2ab+b^2-b}{a^2-a-2ab+b^2+b}=\frac{(a-b)^2+(a-b)}{(a-b)^2-(a-b)}=\frac{a-b+1}{a-b-1},$$

Ans.

6. Multiplying both numerator and denominator by abc, we have

$$\frac{\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}}{\frac{ab}{c}+\frac{ac}{b}+\frac{bc}{a}}=\frac{a^2+b^2+c^2}{a^2b^2+a^2c^2+b^2c^2},\ \textit{Ans.}$$

7. Multiplying both numerator and denominator by $(c+d)(c-d)$, or its equal c^2-d^2, we have

$$\frac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}=\frac{ac+bc-ad-bd+ac-bc+ad-bd}{ac+bc+ad+bd+ac-bc-ad+bd}=\frac{2ac-2bd}{2ac+2bd}=\frac{ac-bd}{ac+bd},\ \mathit{Ans.}$$

8. Multiplying both numerator and denominator by $(a^2-b^2)(a^2+b^2)$, or its equal a^4-b^4, we have

$$\frac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}=\frac{a^4+2a^2b^2+b^4-(a^4-2a^2b^2+b^4)}{a^4+2a^3b+2a^2b^2+2ab^3+b^4-(a^4-2a^3b+2a^2b^2-2ab^3+b^4)}$$

$$=\frac{4a^2b^2}{4a^3b+4ab^3}=\frac{ab}{a^2+b^2},\ \mathit{Ans.}$$

9. The least common multiple of the denominators of the fractional parts is $(x^2-1)(y^2-1)$. Therefore,

$$\frac{\dfrac{1}{x-1}+\dfrac{1}{x+1}}{\dfrac{1}{y-1}+\dfrac{1}{y+1}}=\frac{xy^2-x+y^2-1+xy^2-x-y^2+1}{x^2y-y+x^2-1+x^2y-y-x^2+1}$$

$$=\frac{2xy^2-2x}{2x^2y-2y}=\frac{xy^2-x}{x^2y-y}=\frac{x}{y}\left(\frac{y^2-1}{x^2-1}\right),\ \mathit{Ans.}$$

10. $\dfrac{a+1}{a}+\dfrac{b+1}{b}-\dfrac{c+1}{c}-\dfrac{d+1}{d}=1+\dfrac{1}{a}+1+\dfrac{1}{b}-1-\dfrac{1}{c}-1-\dfrac{1}{d}.$

Hence the numerator $=\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}-\dfrac{1}{d}.$

Substituting this expression for the numerator in the complex fraction, and multiplying by $abcd(c+d)(a+b)$, we have

$$\frac{\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}-\dfrac{1}{d}}{\dfrac{cd}{c+d}-\dfrac{ab}{a+b}}=\frac{(c+d)(a+b)(bcd+acd-abd-abc)}{abcd(a+b)cd-abcd(c+d)ab}$$

$$=\frac{(c+d)(a+b)(bcd+acd-abd-abc)}{abcd(bcd+acd-abd-abc)}$$

$$=\frac{(c+d)(a+b)}{abcd},\ \mathit{Ans.}$$

SIMPLE EQUATIONS.

(**151**, page 88.)

1. Given, $\frac{x}{2}+\frac{2x}{3}-\frac{3x}{4}=10$;

multiplying by 12, $6x+8x-9x=120$, *Ans.*

2. Given, $\frac{3x}{7}-\frac{2x+3}{14}=\frac{x-5}{21}$;

multiplying by 42, $18x-6x-9=2x-10$, *Ans.*

3. Given, $\frac{a}{x-a}+\frac{c}{x+a}=\frac{d}{x^2-a^2}$;

multiplying by (x^2-a^2), $ax+a^2+cx-ca=d$, *Ans.*

4. Given, $\frac{x-a}{c}-\frac{2x-3a}{ac^2}=\frac{x+ac}{a^2}$;

multiplying by a^2c^2, $a^2cx-a^2c-2ax+3a^2=c^2x+ac^2$, *Ans.*

5. Given, $\frac{ax-bx}{8c}-\frac{cx-ax}{10a}=\frac{bx-cx}{4ac}$;

multiplying by $40ac$, $5a^2x-5abx-4c^2x+4acx=10bx-10cx$, *Ans.*

6. Given, $\frac{5x}{12}-\frac{3x}{16}+\frac{3-x}{24}-\frac{5x-2}{20}=2$;

multiplying by 240, $100x-45x+30-10x-60x+24=480$, *Ans.*

7. Given, $\frac{1}{abc}=\frac{a}{bcx}+\frac{b}{acx}+\frac{c}{abx}$;

multiplying by $abcx$, $x=a^2+b^2+c^2$, *Ans.*

REDUCTION OF SIMPLE EQUATIONS.

(**156**, page 92.)

1. Given, $7x-16=3x-4$;
transposing and uniting, $4x=12$;
whence, by division, $x=3$, *Ans.*

2. Given, $3x+9=5x+1$;
transposing and uniting, $-2x=-8$;
dividing by -2, $x=4$, *Ans.*

3. Given, $4x+7=x+21-3+x$;
transposing and uniting, $2x=11$;
$x=5\frac{1}{2}$, *Ans.*

4. Given, $5x+16=x+52$;
transposing and uniting, $4x=36$;
$x=9$, *Ans.*

5. Given, $5ax-c=b-3ax$;
transposing and uniting, $8ax=b+c$;
dividing by $8a$, $x=\frac{b+c}{8a}$, *Ans.*

6. Given, $ax+b=9x+c$;
transposing and factoring, $x(a-9)=c-b$;
dividing by $(a-9)$, $x=\frac{c-b}{a-9}$, *Ans.*

7. Given, $\frac{x}{4}+\frac{x}{6}=10$;
clearing of fractions, $3x+2x=10\times 12$;
uniting, $5x=10\times 12$;
dividing by 5, $x=\ 2\times 12$;
$x=24$, *Ans.*

8. Given, $\frac{3x}{2}=\frac{x}{4}+24;$

clearing of fractions, $6x=x+96;$
transposing and uniting, $5x=96;$
$x=19\frac{1}{5}$, *Ans.*

9. Given, $\frac{3x+5}{2}=\frac{15x-1}{8};$

clearing of fractions, $12x+20=15x-1;$
transposing and uniting, $-3x=-21;$
whence, $x=7$, *Ans.*

10. Given, $\frac{x+1}{3}+\frac{3x-5}{5}=\frac{9x}{10};$

clearing of fractions, $10x+10+18x-30=27x;$
transposing and uniting, $x=20$, *Ans.*

11. Given, $\frac{2x+1}{2}+\frac{7x-15}{5}=\frac{17x+3}{8}-\frac{3}{2};$

clearing of fractions, $40x+20+56x-120=85x+15-60;$
transposing and uniting, $11x=55;$
$x=5$, *Ans.*

12. Given, $\frac{x}{2}+\frac{x}{3}+\frac{5x}{12}=\frac{5x}{7}+\frac{3x}{4}-18;$

clearing of fractions, $42x+28x+35x=60x+63x-18\times 84;$
transposing and uniting, $-18x=-18\times 84;$
dividing by (-18), $x=84$, *Ans.*

13. Given, $\frac{17x-12}{3}-\frac{5x-16}{4}-\frac{10x-3}{6}=\frac{6x-7}{2};$

clearing of fractions, $68x-48-15x+48-20x+6=36x-42;$
transposing and uniting, $-3x=-48;$
whence, by division, $x=16$, *Ans.*

14. Given, $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2};$

clearing of fractions, $336+3x-11=10x-10+776-56x;$
transposing and uniting, $49x=441;$
$x=9$, *Ans.*

15. Given, $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}$;

multiplying by 21, $7x+16-\frac{21x+168}{4x-11}=7x$;

transposing and uniting, $16=\frac{21x+168}{4x-11}$;

clearing of fractions, $64x-176=21x+168$;

whence, $43x=344$;

and $x=8$, *Ans.*

16. Given, $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$;

multiplying by 36, $9x+20=\frac{144x-432}{5x-4}+9x$;

dropping $9x$, $20=\frac{144x-432}{5x-4}$;

clearing of fractions, $100x-80=144x-432$;

transposing and uniting, $-44x=-352$;

whence, $x=8$, *Ans.*

17. Given, $\frac{20x}{25}+\frac{36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+\frac{86}{25}$;

dropping $\frac{20x}{25}+\frac{36}{25}$ $\frac{5x+20}{9x-16}=\frac{50}{25}=2$;

clearing of fractions, $5x+20=18x-32$;

transposing and uniting, $-13x=-52$;

Whence, $x=4$, *Ans.*

18. Given, $\frac{3x}{4}-\frac{x-1}{2}=6x-\frac{20x+13}{4}$;

clearing of fractions, $3x-2x+2=24x-20x-13$;

transposing, &c., $-3x=-15$;

dividing by (-3), $x=5$, *Ans.*

19. Given, $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x+19}{2}$;

clearing of fractions, $3x-9+2x=120-3x-57$;

transposing, &c., $8x=72$;

whence, $x=9$, *Ans.*

20. Given, $\dfrac{x+1}{2}+\dfrac{x+2}{3}=16-\dfrac{x+3}{4}$;

clearing of fractions, $6x+6+4x+8=192-3x-9$;

transposing, &c., $13x=169$;

whence, $x=13$, *Ans.*

21. Given, $2x-\dfrac{x+3}{3}+15=\dfrac{12x+26}{5}$;

clearing of fractions, $30x-5x-15+225=36x+78$;

transposing, &c., $-11x=-132$;

dividing by (-11), $x=12$, *Ans.*

22. Given, $\dfrac{5x+5}{x+2}-29=\dfrac{6x-12}{x-2}-30$;

dropping -29, $\dfrac{5x+5}{x+5}=\dfrac{6x-12}{x-2}-1$;

reducing $\dfrac{6x-12}{x-2}-1$ to the form of a fraction, $\dfrac{5x+5}{x+2}=\dfrac{5x-10}{x-2}$;

clearing of fractions, $5x^2-5x-10=5x^2-20$;

dropping $5x^2$, transposing and uniting, $-5x=-10$;

$x=2$, *Ans.*

23. Given, $\dfrac{7x+9}{4}-\left(x-\dfrac{2x-1}{9}\right)=7$;

removing the parenthesis, $\dfrac{7x+9}{4}-x+\dfrac{2x-1}{9}=7$;

clearing of fractions, $63x+81-36x+8x-4=252$;

transposing and uniting. $35x=175$;

$x=5$, *Ans.*

24. Given, $\dfrac{7+9x}{4}-\left(1-\dfrac{2-x}{9}\right)=7x$;

removing the parenthesis, $\dfrac{7+9x}{4}-1+\dfrac{2-x}{9}=7x$;

clearing of fractions, $63+81x-36+8-4x=252x$;

transposing and uniting, $-175x=-35$;

dividing by (-175), $x=\frac{1}{5}$, *Ans.*

25. Given, $\frac{x+1}{2}+\frac{x+2}{3}=\frac{x-3}{4}+\frac{x-4}{6}+3$;

clearing of fractions, $6x+6+4x+8=3x-9+2x-8+36$;
transposing and uniting, $5x=5$; and $x=1$, *Ans.*

26. Given, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=77$;

clearing of fractions, $30x+20x+15x+12x=77\times 60$;
uniting, $77x=77\times 60$;
whence, by division, $x=60$, *Ans.*

27. Given, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=130$;

clearing of fractions, $6x+4x+3x=130\times 12$;
uniting, $13x=130\times 12$;
$x=10\times 12$;
$x=120$, *Ans.*

28. Given, $\frac{x}{2}+\frac{x}{6}+\frac{x}{12}=90$;

clearing of fractions, $6x+2x+x=90\times 12$;
$9x=90\times 12$;
$x=10\times 12$;
$x=120$, *Ans.*

29. Given, $\frac{x}{2}+\frac{x}{3}+\frac{x}{7}=82$;

clearing of fractions, $21x+14x+6x=82\times 42$;
uniting, $41x=82\times 42$;
$x=2\times 42=82$, *Ans.*

30. Given, $\frac{x}{5}+\frac{x}{7}+\frac{x}{12}+\frac{x}{20}+\frac{x}{21}=660$;

clearing of fractions, $84x+60x+35x+21x+20x=660\times 420$;
uniting, $220x=660\times 420$;
dividing by 220, $x=3\times 420$;
whence, $x=1260$, *Ans.*

31. Given, $a^2x+2ac-c^2x=a^2+c^2$;

transposing, $a^2x-c^2x=a^2-2ac+c^2$:

dividing by (a^2-c^2), and factoring, $x=\frac{(a-c)(a-c)}{(a-c)(a+c)}$;

or, $x=\frac{a-c}{a+c}$, *Ans.*

32. Given, $4bx-2a=3ab-6b^2x$;

transposing, $6b^2x+4bx=3ab+2a$;

whence, $x=\frac{3ab+2a}{6b^2+4b}$;

reducing, $x=\frac{a}{2b}$, *Ans.*

33. Given, $a(x-b)+b(x-c)+c(x-a)=0$;

removing parentheses, $ax-ab+bx-bc+cx-ac=0$;

transposing, $ax+bx+cx=ab+bc+ac$;

factoring, $x(a+b+c)=ab+bc+ac$;

whence, $x=\frac{ab+bc+ac}{a+b+c}$, *Ans.*

34. Given, $a^2(x-1)+am(x-2)=m^2$;

removing parentheses, $a^2x-a^2+amx-2am=m^2$;

transposing and factoring, $a(a+m)x=a^2+2am+m^2$;

whence, $x=\frac{a+m}{a}$, *Ans.*

35. Given, $ax+cx+x=b+\frac{b-ax}{c}$;

clearing of fractions, $acx+c^2x+cx=bc+b-ax$;

transposing, $acx+ax+c^2x+cx=bc+b$;

factoring, $ax(c+1)+cx(c+1)=b(c+1)$;

dividing by $(c+1)$, $ax+cx=b$;

whence, $x=\frac{b}{a+c}$, *Ans.*

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36. Given, $$\frac{a+x}{b}+\frac{c-x}{d}=\frac{a}{b};$$

clearing of fractions, $$ad+dx+bc-bx=ad;$$
transposing, $$-bx+dx=-bc;$$
or by (**150**, III.) $$bx-dx=bc;$$
whence, $$x=\frac{bc}{b-d}, \textit{ Ans.}$$

37. Given, $$\frac{x}{a-1}+\frac{x}{b-1}-\frac{x}{a+1}-\frac{x}{b+1}=1.$$

Observing that $\frac{x}{a-1}-\frac{x}{a+1}=\frac{2x}{a^2-1}$, and $\frac{x}{b-1}-\frac{x}{b+1}=\frac{2x}{b^2-1}$,

we have $$\frac{2x}{a^2-1}+\frac{2x}{b^2-1}=1;$$

clearing of fractions, $$2b^2x-2x+2a^2x-2x=(a^2-1)(b^2-1);$$
uniting and factoring, $$2(b^2-2+a^2)x=(a^2-1)(b^2-1);$$
$$x=\frac{(a^2-1)(b^2-1)}{2(b^2-2+a^2)}, \textit{ Ans.}$$

38. Given, $$\frac{x-1}{c-1}+\frac{x}{c+1}=\frac{1}{c-1}+\frac{2}{(c-1)^2};$$

multiplying by $(c-1)$, $$x-1+\frac{cx-x}{c+1}=1+\frac{2}{c-1};$$

uniting, $$x-2+\frac{cx-x}{c+1}=\frac{2}{c-1};$$

clearing of fractions, $$c^2x-x-2c^2+2+c^2x-2cx+x=2c+2;$$
transposing and uniting, $$2c^2x-2cx=2c^2+2c;$$
dividing by $2c$, and factoring, $$(c-1)x=c+1;$$
whence, $$x=\frac{c+1}{c-1}, \textit{ Ans.}$$

39. Given, $$\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=ab+ac+bc;$$

clearing of fractions, $$x(bc+ac+ab)=(ab+ac+bc)abc;$$
whence, $$x=abc, \textit{ Ans.}$$

40. Given, $$\frac{x-b-c}{a}+\frac{x-a-c}{b}+\frac{x-a-b}{c}=3;$$

clearing of fractions,

$$bcx-(b+c)bc+acx-(a+c)ac+abx-(a+b)ab=3abc;$$

Transposing,

$$bcx+acx+abx=(b+c)bc+(a+c)ac+(a+b)ab+3abc.$$

Removing the parentheses, and arranging the second member of the equation in three terms with reference to the coefficient of x, we have

$$(bc+ac+ab)x=(abc+a^2c+a^2b)+(b^2c+abc+ab^2)+(bc^2+ac^2+abc),$$

dividing by $(bc+ac+ab)$, $\quad x=a+b+c$, *Ans.*

41. Given, $\quad 1.25x-6.125+.25x=.625x;$
transposing, $\quad 1.25x+.25x-.625x=6.125;$
uniting, $\quad .875x=6.125;$
dividing by .875, $\quad x=7$, *Ans*

42. Given, $\quad 3.164x-4.266=.24x+.08x;$
transposing and uniting, $\quad 2.844x=4.266;$
dividing by 2.844, $\quad x=1.5$, *Ans.*

43. Given, $$\frac{2.4x-.12}{2.8}+\frac{4.6x-3.6}{4}=\frac{.64x-.048}{.7};$$

clearing of fractions, $\quad 2.4x-.12+3.22x-2.52=2.56x-.192;$
transposing and uniting $\quad 3.06x=2.448;$
$\quad x=.8$, *Ans.*

PROBLEMS

PRODUCING EQUATIONS WHICH CONTAIN ONE UNKNOWN QUANTITY.

(**160**, page 97.)

1. Let x represent the number; then by the conditions,

$$(x-6)11=121;$$

whence, $\quad x-6=11;$
$\quad x=17$, *Ans.*

2. Let x=time past; then
$20-x$=time to come.

Now by the conditions of the problem, we have $\frac{x}{3}=\frac{20-x}{2}$;

whence, $2x=60-3x$;
$5x=60$, and $x=12$, *Ans.*

3. Let x represent the number; then by the conditions,

$$x+\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=250;$$

whence, $25x=250\times 12$;
$x=10\times 12=120$, *Ans.*

4. Let x=one part; then
$77-x$=the other part;

and by the conditions of the problem, we have $\frac{x}{7}+\frac{77-x}{3}=15$;

$3x+539-7x=315$;
$-4x=-224$;

whence, $x=56$, one part;
and $77-x=21$, the other part.

5. Let x=the greater number; then
$75-x$= the lesser number;

and by the conditions of the problem, we have $x-(75-x)=\frac{x}{3}$;

$3x-225+3x=x$;
$5x=225$;

whence, $x=45$, the greater;
and $75-x=30$, the less.

6. Let x represent the amount of money at first; then by the conditions, we have

$$x-\frac{x}{4}-\frac{x}{5}=66;$$

$20x-5x-4x=66\times 20$;
$11x=66\times 20$;

whence, $x=6\times 20=120$, *Ans.*

7. Let $30x$ represent my money at first; then

$$30x-\frac{30x}{3}=20x=\text{remainder after the first payment};$$

$$20x-\frac{20x}{4}=15x=\text{remainder after the second payment};$$

whence, $$15x-\frac{15x}{5}=12x=24;$$

$$x=2;$$

$$30x=60, \textit{ Ans.}$$

8. Let x represent the number; then by the conditions,

$$\frac{2}{3}(x-5)=40;$$

$$2x-10=120;$$

$$x=65, \textit{ Ans.}$$

9. Let x=price of the horse; then

$$200-x=\text{price of the chaise};$$

and by the conditions, $$\frac{x}{2}=\frac{1}{3}(200-x);$$

$$3x=400-2x;$$

$$x=80, \text{ the price of the horse,}$$

$$200-x=120, \text{ the price of the chaise.}$$

10. Let x=the greater part; then

$$48-x=\text{the lesser part};$$

and by conditions, $$\frac{x}{6}+\frac{48-x}{4}=9;$$

$$4x+288-6x=216;$$

whence, $$x=36, \text{ the greater};$$

and $$48-x=12, \text{ the lesser.}$$

11. Let x represent the value of the estate; then by the conditions,

$$\frac{x}{4}+200=\text{what the first received,}$$

$$\frac{x}{5}+340=\text{what the second received,}$$

$$\frac{x}{6}+300=\text{what the third received,}$$

$$\frac{x}{8}+400=\text{what the fourth received,}$$

whence, $\frac{x}{4}+\frac{x}{5}+\frac{x}{6}+\frac{x}{8}+1240=x;$

$$30x+24x+20x+15x+1240\times120=120x;$$

$$31x=1240\times120;$$

$$x=40\times120=4800, \textit{ Ans.}$$

12. Let x represent the number; then by the conditions,

$$\frac{1}{3}(x-91)=\frac{x}{10};$$

whence, $10x-910=3x;$

$$7x=910, \text{ and } x=130, \textit{ Ans.}$$

13. Let $x=$ A s stock; then

$3x=$ B's stock;

$12x=$ C's stock;

$5x=$ D's stock.

By the conditions, $21x=$ 73500, and $x=\$3500$, *Ans.*

15. Let $7x=$ A's share; then $9x=$ B's share; and by the conditions,

$$7x+9x=\$2000;$$

whence, $7x=\$875$, A's share; $9x=\$1125$, B's share.

16. Let $3x=$ the rye; then $4x=$ the oats, and $5x=$ the peas. By the conditions,

$$3x+4x+5x=72;$$

whence, $x=6$; $3x=18$; $4x=24$, and $5x=30$;

therefore, Rye, 18 bushels; oats, 24 bushels; peas, 30 bushels, *Ans.*

17. Let $6x=$ the quantity of wine drawn from one cask; then
$7x=$ the quantity of wine drawn from the other.

By the conditions,

$$7x-16=\frac{6x}{2}=3x$$

whence, $x=4$, $7x=28$, and $6x=24$.

18. Let $5x=$ less part, and $a=204$;

then, $a-5x=$ greater part;

whence, $a-7x=20x-\frac{3}{7}(a-5x)$;

$$204x=10a=10\times 204;$$

$$x=10;$$

$5x=50$, the lesser;

$204-5x=154$, the greater.

19. Let $8x=$ the price of the horse; then
$a-8x=$ the price of the chaise.

Now by the conditions,

$$2a-16x-3x=24x-\frac{5}{7}(a-8x);$$

$$148-133x=168x-5a+40x,$$

$$341x=19a=19\times 341;$$

$$x=19;$$

whence, $8x=152$, horse,

and, $341-8x=189$, chaise.

21. Put $7560=a$; then $9560=a+2000$.

Let $x=$ each private's share; then
$27x=$ what all the privates received;
$27x+a=$ the prize.

By conditions of the problem, we have

$$27x+a=25x+a+2000;$$

$$2x=2000;\ x=1000;$$

$27x+7560=34560$, *Ans.*

22. As he annually increases his unexpended capital by one third of itself, four thirds of this unexpended part will represent his capital at the end of each year.

Let $x=$ his original stock; put $1000=a$; then

$$\frac{4}{3}(x-a)=\frac{4x-4a}{3}= \text{stock at the end of the first year;}$$

$$\frac{4}{3}\left(\frac{4x-4a}{3}-a\right)=\frac{16x-28x}{9}= \text{stock at the end of the second year;}$$

$$\frac{4}{3}\left(\frac{16x-28a}{9}-a\right)=\frac{64x-148a}{27}= \text{stock at the end of the third year;}$$

therefore,

$$\frac{64x-148a}{27}=2x;$$

$$64x-148a=54x;$$

$$10x=148a=148\times 1000;$$

$$x=14800, \textit{Ans.}$$

23. Put $a=99$. Let $x=$ the time past; then

$$a-x=\text{the time to come.}$$

Now by the conditions,

$$\frac{2x}{3}=\frac{4}{5}(a-x);$$

$$10x=12a-12x;$$

$$22x=12a=12\times 99;$$

whence,

$$x=6\times 9=54;$$

$$99-x=45, \textit{Ans.}$$

24. Let $6x=$ the amount of gunpowder; then,

$$4x+10=\text{the nitre;}$$

$$x-4\tfrac{1}{2}=\text{the sulphur;}$$

$$\frac{4x+10}{7}-2=\text{the charcoal.}$$

By the conditions,

$$4x+10+x-4\tfrac{1}{2}+\frac{4x-10}{7}-2=6x;$$

$$\frac{4x-10}{7}-x=-\frac{7}{2};$$

$$8x+20-14x=-49;$$

whence,

$$6x=69, \textit{Ans.}$$

25. Put $a=183$. Let $x=$ what the first received; then, $a-x=$ what the second received.

By the conditions,

$$\frac{4x}{7}=\frac{3}{10}(a-x);$$

$$40x=21a-21x;$$

$$61x=21a=21\times 183;$$

whence,

$$\left.\begin{array}{r} x=21\times 3= \quad 63 \\ 183-x= \quad 120 \end{array}\right\} Ans.$$

26. Let $x=$ the greater part; then, $68-x=$ the lesser part.

By the conditions,

$$84-x=3[40-(68-x)];$$

$$84-x=120-204+3x;$$

whence,

$$4x=168,$$

$$\left.\begin{array}{r} x=42 \\ 68-x=26 \end{array}\right\} Ans.$$

27. Let $2x=$ distance from A to B; then $3x=$ distance from C to D;

$$\frac{1}{3}\left(\frac{2x}{4}+\frac{3x}{2}\right)=\frac{2x}{3}= \text{ distance from B to C.}$$

By the conditions of the problem,

$$5x+\frac{2x}{3}=34;$$

$$17x=34\times 3;$$

$$x=6.$$

Whence,

$2x=12$, distance from A to B;

$3x=18$, distance from C to D;

$\frac{2}{3}x=\ 4$, distance from B to C.

28. Let $3x=$ the number of sheep at first; then $2x-6=$ the number after the first plunder. Whence,

$$2x-6-(x-3)-10=2;$$

$$x=15;\ 3x=45,\ Ans.$$

29. Observe that for every vessel broken, he lost 12 cents,—3 cents fee, and 9 cents forfeiture.

Let $x=$ the number he broke; then

$$300-12x=240;$$

$$x=5, \textit{ Ans.}$$

30. Let $x=$ his indebtedness to A; then

$2x=$ his indebtedness to B;

$6x=$ his indebtedness to C.

By the conditions, $9x=270$; and $x=30$.

31. Let $6x=$ A's share, and $9x=$ B's share; then

$20x=$ C's share, and $43\frac{3}{4}x=$ D's share.

By the conditions, $78\frac{3}{4}x=315$; whence,

$$6x=24,\quad 9x=36,\quad 20x=80,\quad 43\tfrac{3}{4}x=175.$$

32. Let $5x=$ his money at first; then

$4x+4=$ what he had after first losing and winning;

$3x+3+3=$ what he had after second losing and winning.

As he loses $\frac{1}{6}$ of this, he must have $\frac{5}{6}$ of it left; therefore

$$\tfrac{5}{6}(3x+6)=20;$$

$$15x+30=120;$$

whence, $15x=90$, and $5x=30$, *Ans.*

33. Let $3x=$ his income; then

$2x=$ his family expenses.

The remainder of his income is x, $\frac{2}{3}$ of which he spends in improvements, leaving $\frac{1}{3}$ of x to lay by; whence,

$$\frac{x}{3}=70;\text{ and } 3x=630, \textit{ Ans.}$$

34. Let $x=$ the lesser part; then

$60-x=$ the greater part.

By the conditions, $x(60-x)=3x^2$;

$$60-x=3x;$$

whence, $4x=60$, and $x=15$, *Ans.*

35. Let $x=$ the value of the saddle; then

$8x=$ the value of the horse.

By the conditions, $9x=90$; whence $x=10$, *Ans.*

36. Let x = what one receives; then
$10x$ = what the other receives.

By the conditions, $11x=462$; whence $x=42$, *Ans.*

37. Let x = rent of the estate last year; then by the conditions,

$$x+\frac{8x}{100}=1890;$$

whence, $108x=189000$, and $x=1750$, *Ans.*

38. Let x = the greater; then
$840-x$ = the less.

By the conditions, $$x-(840-x)=\frac{x}{3};$$

$$3x-2520+3x=x;$$

whence, $5x=2520$, and $x=504$, *Ans.*

39. Let x = his income; then by the conditions,

$$x-\left(\frac{x}{5}+100\right)=\frac{x}{2}+35;$$

$$10x-2x-1000=5x+350;$$

whence, $3x=1350$, and $x=450$, *Ans.*

40. Let x = A's part; then
$100+x$ = B's part.
$370+x$ = C's part.

By the conditions, $3x+470=1520$;
whence, $x=350$, A's part.

41. Let $7x$ = the income; then
x = A's annual debt;
$\frac{7x}{5}$ = what B saves annually.

By the conditions, $$2\left(\frac{7x}{5}\right)=2x+32;$$

$$\frac{7x}{5}=x+16;$$

whence, $x=40$;
$7x=280$, *Ans.*

42. A's rate of travel is $\frac{7}{5}$ miles per hour;

B's rate of travel is $\frac{5}{3}$ miles per hour;

$\frac{56}{5}$ = the distance between A and B, when B sets out.

Let x = the number of hours; then by the conditions,

$$\frac{5x}{3}=\frac{7x}{5}+\frac{56}{5};$$

$$25x=21x+168;$$

whence, $x=42$, number of hours;

$$\frac{5x}{3}=70, \text{ number of miles.}$$

43. Let x = the number of days; then both working together will do $\frac{1}{x}$ of the work in one day. But A does $\frac{1}{8}$ of the work in one day, and B $\frac{1}{12}$; therefore,

$$\frac{1}{8}+\frac{1}{12}=\frac{1}{x};$$

$$\frac{x}{8}+\frac{x}{12}=1;$$

whence, $3x+2x=24$, and $x=4\frac{4}{5}$, *Ans.*

44. Let x = the distance; then

$\frac{x}{8}$ = hours he rides;

$\frac{x}{4}$ = hours he walks.

By the conditions, $\frac{x}{8}+\frac{x}{4}=6$;

whence, $3x=48$, and $x=16$, *Ans.*

45. Let $3x$ = the time in which C can dig the trench;

then $2x$ = the time in which B can dig the trench;

and x = the time in which A can dig the trench.

Now in one day A can dig $\frac{1}{x}$ of it, B $\frac{1}{2x}$ of it, and C $\frac{1}{3x}$ of it; but all together can dig $\frac{1}{6}$ of it in one day. Hence,

$$\frac{1}{6}=\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x};$$

$$x=11, \quad \text{A's time,}$$
$$2x=22, \quad \text{B's time,}$$
$$3x=33, \quad \text{C's time.}$$

46. Let $x=$ the distance between A and B. Since C and B travel at the respective rates of 5 and 4 miles an hour, when they have met, C has traveled $\frac{5}{9}$ of the whole distance, or $\frac{5x}{9}$ miles. In the meantime, A, traveling $\frac{3}{5}$ as fast as C, has traveled $\frac{3}{5}$ of $\frac{5x}{9}$, or $\frac{x}{3}$ miles. Hence, $\frac{5x}{9}-\frac{x}{3}=\frac{2x}{9}$, the distance between A and C when C turns back.

Now C in traveling back to meet A, goes $\frac{5}{8}$ of this distance, or $\frac{5}{8}$ of $\frac{2x}{9}=\frac{5x}{36}$ miles. But the *whole* distance traveled by C is 50 miles; hence,

$$\frac{5x}{9}+\frac{5x}{36}=50;$$

$$x=72, \textit{ Ans.}$$

47. Let $x=$ the value of one sheep; then

$$72x+92x=164x= \text{ the value of the flock.}$$

By the conditions,

$$92x-\$35=72x+\$35,$$

$$x=\$3\tfrac{1}{2}; \text{ whence, } 164x=\$574, \textit{ Ans.}$$

48. Let $x=$ the rate of the current per hour. As the current will retard the boat by its whole velocity in going up the river, and accelerate it the same in going down, $12-x=$ actual rate of rowing up stream, and $12+x=$ actual rate of rowing down stream. By the conditions,

$$7(12-x)=5(12+x);$$
$$84-7x=60+5x;$$
$$12x=24;$$

whence,

$$x=2, \textit{ Ans.}$$

SIMPLE EQUATIONS

CONTAINING TWO UNKNOWN QUANTITIES.

(**170**, page 109.)

1. Given, $\begin{cases} 8x+5y=68, & (1) \\ 12x+7y=100; & (2) \end{cases}$

multiplying equation (1) by 3, and equation (2) by 2, we have

$$24x+15y=204; \quad (3)$$
$$24x+14y=200; \quad (4)$$

subtracting (4) from (3), $y=4$;
substituting this value of y in (1), $8x+20=68$;
whence, $8x=48$, and $x=6$, *Ans.*

2. Given, $\begin{cases} 5x+2y=79, & (1) \\ 7x-6y=9; & (2) \end{cases}$

multiplying (1) by 3, $15x+6y=57$; (3)
adding (2) and (3), $22x=66$;
whence, $x=3$, and $y=2$, *Ans.*

3. Given, $\begin{cases} 3x+7y=79, & (1) \\ x+4y=38; & (2) \end{cases}$

multiplying (2) by 3, $3x+12y=114$; (3)
subtracting (1) from (3), $5y=35$;
whence, $y=7$, and $x=10$, *Ans.*

4. Given, $\begin{cases} 5x-3y=36, & (1) \\ 2x+9y=96; & (2) \end{cases}$

multiplying (1) by 3, $15x-9y=108$; (3)
adding (2) and (3), $17x=204$;
whence, $x=12$, and $y=8$, *Ans.*

5. Given, $\begin{cases} x+17y=54, & (1) \\ 3x-25y=10; & (2) \end{cases}$

multiplying (1) by 3, $3x+51y=162$; (3)
subtracting (2) from (3), $76y=152$;
whence, $y=2$, and $x=20$, *Ans.*

6. Given, $5x-4y=\ 40,$ (1)
$x-5y=-97;$ (2)

multiplying (2) by 5, $5x-25y=-485;$ (3)
subtracting (3) from (1), $21y=\ 525;$
whence, $y=\ 25$, and $x=28$, *Ans.*

7. Given, $\begin{cases} 8x+15y=\ 9, & (1) \\ 6x-12y=-1; & (2) \end{cases}$

multiplying (1) by 3, $24x+45y=\ 27;$ (3)
multiplying (2) by 4, $24x-48y=-4;$ (4)
subtracting (4) from (3), $93y=\ 31;\ y=\frac{31}{93}=\frac{1}{3}.$
substituting the value of y in (1), $8x+5=\ 9;\ x\ =\frac{1}{2}.$ *Ans.*

8. Given, $\begin{cases} 7x+7y=30, & (1) \\ 3x+4y=17; & (2) \end{cases}$

multiplying (1) by 3, $21x+21y=90;$ (3)
multiplying (2) by 7, $21x+28y=119;$ (4)
subtracting (3) from (4), $7y=29;\ y=4\frac{1}{7},$
substituting the value of $7y$ in (1), $7x+29=30;\ x=\ \frac{1}{7}.$ *Ans.*

9. Given, $\begin{cases} 8x+3y=25, & (1) \\ 5x-6y=55; & (2) \end{cases}$

multiplying (1) by 2, $16x+6y=50;$ (3)
adding (2) and (3), $21x=105,$
whence, $x=5$, and $y=-5$, *Ans.*

10. Given, $\begin{cases} 15x-8y=\ 9, & (1) \\ 10x+4y=-43. & (2) \end{cases}$

multiplying (2) by 2, $20x+8y=-86;$ (3)
adding (1) and (3), $35x=-77;$
whence, $x=-2\frac{1}{5}$, and $y=-5\frac{1}{4}$, *Ans.*

11. Given, $\begin{cases} 9x-5y=\ 950, & (1) \\ 2x-3y=-450; & (2) \end{cases}$

multiplying (1) by 3, $27x-15y=\ 2850;$ (3)
multiplying (2) by 5, $10x-15y=-2250;$ (4)
snbtracting (4) from (3), $17x=\ 5100;$
whence, $x=300$, and $y=350$, *Ans.*

12. Given, $\begin{cases} \dfrac{x}{2}-\dfrac{y}{4}=20, & (1) \\ \dfrac{x}{12}+\dfrac{y}{8}=10\,; & (2) \end{cases}$

multiplying (2) by 2, $\dfrac{x}{6}+\dfrac{y}{4}=20\,;$ (3)

adding (1) and (3), $\dfrac{2x}{3}=40\,;$

whence, $2x=120\,;\ x=60.$

substituting in (1), $30-\dfrac{y}{4}=20\,;\ \ y=40.$ } *Ans.*

13. Given, $\begin{cases} \dfrac{x}{2}+\dfrac{y}{3}=\ \ 8, & (1) \\ \dfrac{x}{3}-\dfrac{y}{5}=-1\,; & (2) \end{cases}$

multiplying (1) by 2, $x+\dfrac{2y}{3}=\ \ 16\,;$ (3)

multiplying (2) by 3, $x-\dfrac{3y}{5}=-3\,;$ (4)

subtracting (4) from (3), $\dfrac{2y}{3}+\dfrac{3y}{5}=\ \ 19\,;$ (5)

clearing (5) of fractions, $10y+9y=19\times 15\,;$
whence, $y=15\,;$
substituting this value of y in (3), $x+10=16\,;$
$x=6.$

14. Given, $\begin{cases} 3x-\dfrac{y}{2}=3\tfrac{1}{2}, & (1) \\ 4y-\dfrac{y}{5}=7\,; & (2) \end{cases}$

multiplying (1) by 2, $6x-y=\ \ 7\,;$ (3)
multiplying (2) by 5, $20x-y=35\,;$ (4)
subtracting (3) from 4, $14x=28\,;$
whence, $x=2$, and $y=5$, *Ans.*

15. Given,
$$\begin{cases} \dfrac{x}{8}+8y=194, & (1) \\ \dfrac{y}{8}+8x=131; & (2) \end{cases}$$

adding (1) and (2), $\dfrac{x+y}{8}+8(x+y)=325;$ (3)

clearing (3) of fractions, $x+y+64(x+y)=325\times 8;$ (4)

or, $65(x+y)=325\times 8;$ (5)

$x+y=\ \ 5\times 8;$ (6)

clearing (1) of fractions, $x+64y=194\times 8;$ (7)

subtracting (6) from (7), $63y=189\times 8;$

whence, $y=24$, and $x=16$, *Ans.*

16. Given,
$$\begin{cases} \dfrac{x}{3}+3y=21, & (1) \\ \dfrac{y}{3}+3x=29; & (2) \end{cases}$$

adding (1) and (2), $\frac{1}{3}(x+y)+3(x+y)=50;$ (3)

whence, $10(x+y)=50\times 3;$ (4)

or, $x+y=\ 5\times 3;$ (5)

from (1), $x+9y=21\times 3;$ (6)

subtracting (5) from (6), $8y=16\times 3;$

whence, $y=6$, and $x=9$, *Ans.*

17. Given,
$$\begin{cases} \dfrac{x}{7}+7y=99, & (1) \\ \dfrac{y}{7}+7x=51; & (2) \end{cases}$$

adding (1) and (2), $\frac{1}{7}(x+y)+7(x+y)=150;$ (3)

whence, $50(x+y)=150\times 7;$ (4)

or, $x+y=\ \ 3\times 7;$ (5)

from (1), $x+49y=\ 99\times 7;$ (6)

subtracting (5) from (6), $48y=\ 96\times 7;$

whence, $y=14$, and $x=7$, *Ans.*

18. Given, $\begin{cases} \dfrac{4}{x}-\dfrac{4}{y}=1, & (1) \\ \dfrac{4}{x}-\dfrac{2}{y}=1\frac{1}{2}\,; & (2) \end{cases}$

subtracting (2) from (1), $\dfrac{2}{y}=\dfrac{1}{2}$, and $y=4$

substituting value of y in (1), $\dfrac{4}{x}=2$, and $x=2$ } *Ans.*

19. Given, $\begin{cases} \dfrac{147}{x}-\dfrac{147}{56}=28, & (1) \\ \dfrac{17}{x}+\dfrac{56}{y}=13\frac{2}{3}\,; & (2) \end{cases}$

dividing (1) by 147, $\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{4}{21}$; (3)

multiplying (3) by 17, $\dfrac{17}{x}-\dfrac{17}{y}=\dfrac{68}{21}$; (4)

subtracting (4) from (2), $\dfrac{73}{y}=\dfrac{41}{3}-\dfrac{68}{21}=\dfrac{219}{21}$; (5)

dividing (5) by 73, $\dfrac{1}{y}=\dfrac{3}{21}=\dfrac{1}{7}$, and $y=7$

substituting value of y in (3), $\dfrac{1}{x}=\dfrac{7}{21}=\dfrac{1}{3}$, and $x=3$ } *Ans.*

20. Given, $\begin{cases} \dfrac{4x+17}{4}=x+\dfrac{68}{x+y}\,; & (1) \\ \dfrac{5y+27}{5}=y+\dfrac{54}{x-y}. & (2) \end{cases}$

Reducing the first members of equations (1) and (2) to mixed quantities, we have

$$x+\frac{17}{4}=x+\frac{68}{x+y}\,; \qquad (3)$$

$$y+\frac{27}{5}=y+\frac{54}{x-y}\,; \qquad (4)$$

dropping x in (3), and dividing by 17, $\quad \frac{1}{4}=\frac{4}{x+y};$ (5)

dropping y in (4), and dividing by 27, $\quad \frac{1}{5}=\frac{2}{x-y};$ (6)

clearing (5) of fractions, $\quad x+y=16;$ (7)

clearing (6) of fractions, $\quad x-y=10;$ (8)

adding (7) and (8), $\quad 2x=26$, and $x=13$

subtracting (8) from (7), $\quad 2y=6$, and $y=3$ } *Ans.*

21. Given,
$$\begin{cases} x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2}, & (1) \\ y+\frac{y-3}{x-18}=30-\frac{73-3y}{3}; & (2) \end{cases}$$

multiplying (1) by 2, $\quad 2x-\frac{4y-2x}{23-x}=40-59+2x,$ (3)

or, $\quad 19=\frac{4y-2x}{23-x};$ (4)

clearing (4) of fractions, $\quad 437-19x=4y-2x,$ (5)

or, $\quad 17x+4y=437;$ (6)

multiplying (2) by 3, $\quad 3y+\frac{3y-9}{x-18}=90-73+3y,$ (7)

or, $\quad \frac{3y-9}{x-18}=17;$ (8)

clearing (8) of fractions, $\quad 3y-9=17x-306;$ (9)

transposing, $\quad 17x-3y=297;$ (10)

subtracting (10) from (6), $\quad 7y=140$, and $y=20$

whence, from (10), $\quad 17x=357$, and $x=21$ } *Ans.*

22. Given,
$$\begin{cases} \frac{6x^2-24y^2+130}{2x-4y+3}=3x+6y+1, & (1) \\ \frac{9xy-110}{3y-4}+\frac{151-16x}{4y-1}=3x; & (2) \end{cases}$$

clearing (1) of fractions, $\quad 6x^2-24y^2+130=6x^2-24y^2+11x+14y+3;$ (3)

whence, $\quad 11x+14y=127;$ (4)

multiplying (2) by $(3y-4)$, $\quad 9xy-110+\dfrac{(151-16x)(3y-4)}{4y-1}=9xy-12x;$ (5)

whence, $\quad 110-12x=\dfrac{(151-16x)(3y-4)}{4y-1};$ (6)

or, $\quad 440y-48xy-110+12x=453y-48xy-604+64x;$ (7)

or, $\quad 52x+13y=498;$ (8)

or, $\quad 4x+y=38;$ (9)

adding (4) and (9), $\quad 15x+15y=165;$ (10)

$\quad x+y=11;$ (11)

subtracting (11) from (9), $\quad 3x=27$, and $x=9$

substituting value of x in (9), $\quad y=2$ *Ans.*

23. Given, $\begin{cases} \dfrac{x+3y}{3}-\dfrac{7x-21}{6}=\dfrac{3x-15}{4}-\dfrac{8x-9y}{12}, & (1) \\ \dfrac{2x+y}{2}-\dfrac{9x-7}{8}=\dfrac{3y+9}{4}-\dfrac{4x+5y}{16}; & (2) \end{cases}$

clearing (1) of fractions, $\quad 4x+12y-14x+42=9x-45-8x+9y,$ (3)

transposing and uniting, $\quad 11x-3y=87;$ (4)

clearing (2) of fractions, $\quad 16x+8y-18x+14=12y+36-4x-5y;$ (5)

whence, $\quad 2x+y=22;$ (6)

multiplying (6) by 3, $\quad 6x+3y=66;$ (7)

adding (4) and (7), $\quad 17x=153$, and $x=9$

substituting value of x in (6), $\quad 18+y=\ 22$, and $y=4$ *Ans.*

24. Given, $\begin{cases} ax+by=d, & (1) \\ a'x+b'y=d'; & (2) \end{cases}$

multiplying (1) by b', $\quad b'ax+b'by=b'd;$ (3)

" (2) by b, $\quad ba'x+b'by=bd';$ (4)

subtracting (4) from (3), $\quad (b'a-ba')x=b'd-bd'$ (5)

whence, $\quad x=\dfrac{b'd-bd'}{b'a-ba'};$

multiplying (1) by a', $\quad a'ax+a'by=a'd;$ (6)

" (2) by a, $\quad a'ax+ab'y=ad';$ (7)

subtracting (7) from (6), $\quad (a'b-ab')y=a'd-ad';$ (8)

whence, $\quad y=\dfrac{a'd-ad'}{a'b-ab'}.$

25. Given,
$$\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=2ab, & (1) \\ \dfrac{x}{ab}+\dfrac{y}{ab}=a+b\,; & (2) \end{cases}$$

multiplying (2) by b, $\quad \dfrac{x}{a}+\dfrac{y}{a}=ab+b^2\,;$ (3)

subtracting (3) from (1), $\quad \dfrac{y}{b}-\dfrac{y}{a}=ab-b^2=b(a-b)\,;$ (4)

clearing (4) of fractions, $\quad y(a-b)=ab^2(a-b)$, and $y=ab^2$

whence, from (1), $\quad \dfrac{x}{a}+ab=2ab,$ and $x=a^2b$ *Ans.*

26. Given,
$$\begin{cases} ax+cy=\dfrac{a^4+c^4}{a^2c^2}, & (1) \\ cx+ay=\dfrac{a^2+c^2}{ac}\,; & (2) \end{cases}$$

multiplying (1) by a, $\quad a^2x+acy=\dfrac{a^4+c^4}{ac^2}\,;$ (3)

multiplying (2) by c, $\quad c^2x+acy=\dfrac{a^2+c^2}{a}\,;$ (4)

subtracting (4) from (3), $\quad (a^2-c^2)x=\dfrac{a^4+c^4}{ac^2}-\dfrac{a^2+c^2}{a}=\dfrac{a^2(a^2-c^2)}{ac^2}\,;$

whence, $$x=\frac{a^2(a^2-c^2)}{ac^2(a^2-c^2)}=\frac{a^2}{ac^2}$$

or, $$x=\frac{a}{c^2}\,;$$

substituting value of x in (2), $\quad \dfrac{a}{c}+ay=\dfrac{a^2+c^2}{ac}\,;$

whence, $$ay=\frac{a^2+c^2}{ac}-\frac{a}{c}=\frac{c^2}{ac}=\frac{c}{a},$$

or, $$y=\frac{c}{a^2}.$$

SIMPLE EQUATIONS

CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

(172, page 116.)

1. Given,
$$\begin{cases} 2x+4y-3z=22, & (1) \\ 4x-2y+5z=18, & (2) \\ 6x+7y-\ z=63. & (3) \end{cases}$$

As the coefficients of x in (2) and (3) are multiples of the coefficient of x in (1), combine (1) with (2), and then with (3), to eliminate x.

Multiplying (1) by 2,	$4x+8y-6z=44,$	(4)
bringing down (2),	$4x-2y+5z=18;$	
by subtraction,	$10y-11z=26;$	(5)
multiplying (1) by 3,	$6x+12y-9z=66;$	(6)
bringing down (3),	$6x+7y-z=63;$	
by subtraction,	$5y-8z=\ 3;$	(7)
multiplying (7) by 2,	$10y-16z=\ 6;$	(8)
subtracting (8) from (5),	$5z=20$, and $z=4$;	
substituting value of z in (7),	$5y=35$, and $y=7$;	
" values of z and y in (1),	$x=3$.	

2. Given,
$$\begin{cases} 3x+9y+8z=41, & (1) \\ 5x+4y-2z=20, & (2) \\ 11x+7y-6z=37; & (3) \end{cases}$$

multiplying (2) by 4,	$20x+16y-8z=\ 80;$	(4)
bringing down (1),	$3x+9y+8z=\ 41;$	
by addition,	$23x+25y=121;$	(5)
multiplying (2) by 3,	$15x+12y-6z=\ 60;$	(6)
bringing down (3),	$11x+7y-6z=\ 37;$	
by subtraction,	$4x+5y=\ 23;$	(7)
multiplying (7) by 5,	$20x+25y=115;$	(8)
subtracting (8) from (5),	$3x=\ 6$, and $x=2$;	
whence, by (8),	$25y=75$, and $y=3$;	
and by (1),	$8z=\ 8$, and $z=1$.	

3. Given,
$$\begin{cases} x+y+z=32, & (1) \\ x+y-z=25, & (2) \\ x-y-z=\ 9\ ; & (3) \end{cases}$$

adding (1) and (3), $2x=40$, and $x=20$;
subtracting (3) from (2), $2y=16$, and $y=\ 8$;
" (2) " (1), $2z=\ 6$, and $z=\ 3$.

4. Given,
$$\begin{cases} x+y+z=26, & (1) \\ x-y\quad =\ 4, & (2) \\ x-z\quad =\ 6\ ; & (3) \end{cases}$$

adding the three equations, $3x=36$, and $x=12$;
substituting value of x in (2), $y=\ 8$;
" " " (3), $z=\ 6$.

5. Given,
$$\begin{cases} x-y-z=\ 6, & (1) \\ 3y-x-z=12, & (2) \\ 7z-y-x=24. & (3) \end{cases}$$

Assume $x+y+z=s$;
equation (1) becomes $2x=6+s$, (4)
" (2) " $4y=12+s$, (5)
" (3) " $8z=24+s$. (6)

multiplying (4) by 4, $8x=24+4s$; (7)
" (5) by 2, $8y=24+2s$; (8)
bringing down (6), $8y=24+s$;

by addition, $8s=72+7s$;
whence, $s=72$;
substituting value of s in (4), $x=39$;
" " " (5), $y=21$;
" " " (6), $z=12$.

6. Given,
$$\begin{cases} 2x=u+y+z, & (1) \\ 3y=u+x+z, & (2) \\ 4z=u+x+y, & (3) \\ \ u=x-14\ ; & (4) \end{cases}$$

Assuming $x+y+z+u=s$, and adding x to both sides of (1), y to both sides of (2), and z to both sides of (3),

equation (1) becomes	$3x=s$;	(5)
" (2) "	$4y=s$;	(6)
" (3) "	$5z=s$;	(7)
" (4) "	$u=\frac{s}{3}-14$;	(8)
multiplying (5) by 20,	$60x=20s$;	(9)
" (6) by 15,	$60y=15s$;	(10)
" (7) by 12,	$60z=12s$;	(11)
" (8) by 60,	$60u=20s-840$;	(12)
by addition,	$60s=67s-840$;	
whence,	$s=120$;	
substituting value of s in (5),	$x=40$;	
" " " (6),	$y=30$;	
" " " (7),	$z=24$;	
" " " (8),	$u=26$.	

ANOTHER METHOD.

Subtracting (2) from (1),	$2x-3y=y-x$, or $3x=4y$;	(5)
" (2) " (3),	$4z-3y=y-z$, or $5z=4y$;	(6)
adding (3) and (4),	$4z=2x+y-14$;	(7)
multiplying (7) by 5,	$20z=10x+5y-70$;	(8)
" (6) by 4,	$20z=16y$;	(9)
by subtraction	$10x-11y=70$;	(10)
multiplying (10) by 3,	$30x-33y=210$;	(11)
" (5) by 10,	$30x-40y=0$;	(12)
subtracting (12) from (11),	$7y=210$, and $y=30$;	
from (5), (6), and (4),	$x=40$, $z=24$, $u=26$.	

7. Given,
$$\begin{cases} u+3x-y-z=7, & (1)\\ 2u-2x+y+3z=8, & (2)\\ 3u-x+y-4z=8, & (3)\\ 4u+x-y-2z=7; & (4)\end{cases}$$

adding (1) and (2),	$3u+x+2z=15$;	(5)
" (1) " (3),	$4u+2x-5z=15$;	(6)
" (2) " (4),	$6u-x+z=15$;	(7)
" (5) " (7),	$9u+3z=30$;	(8)
" (3) " (4),	$7u-6z=15$;	(9)

adding (8) and (9), $16u-3z=45$; (10)

" (8) " (10), $25u=75$, and $u=3$;

substituting value of u in (8), $z=1$;

" values of u and z in (5), $x=4$;

" " of u, z and x in (2), $y=7$.

8. Given,
$$\begin{cases} 5x-y+7z=61, & (1)\\ 4x+3y+3z=8, & (2)\\ 3x-y-5z=3; & (3)\end{cases}$$

multiplying (3) by 3, $9x-3y-15z=9$; (4)

adding (4) and (2), $13x-12z=17$; (5)

subtracting (3) from (1), $2x+12z=58$; (6)

adding (5) and (6), $15x=75$, and $x=5$;

substituting value of x in (6), $12z=48$, and $z=4$;

" values of x and z in (1), $y=-8$.

9. Given,
$$\begin{cases} u+v+x+y+2z=52, & (1)\\ u+v+x+z+2y=50, & (2)\\ u+v+y+z+2x=48, & (3)\\ u+x+y+z+2v=46, & (4)\\ v+x+y+z+2u=44. & (5)\end{cases}$$

Adding the equations, $6u+6v+6x+6y+6z=240$; (6)

whence, $u+v+x+y+z=40$; (7)

subtracting (7) from (1), $z=12$;

" (7) " (2) $y=10$;

" (7) " (3), $x=8$;

" (7) " (4), $v=6$;

" (7) " (5), $u=4$.

ANOTHER METHOD.

Assuming $u+v+x+y+z=s$, and subtracting this from each given equation,

equation (1) becomes $z=52-s$; (6)

" (2) " $y=50-s$; (7)

" (3) " $x=48-s$; (8)

" (4) " $v=46-s$; (9)

" (5) " $u=44-s$; (10)

(116—117)

by addition, $s=240-5s;$ (11)
whence, $s=40;$ (12)
substituting value of s in (6), (7), (8), (9) and (10),

$z=12,\ y=10,\ x=8,\ v=6,\ u=4,$ *Ans.*

10. Given,
$$\begin{cases} 2x+y-2z=40, & (1)\\ 4y-x+3z=35, & (2)\\ 3u+t=13, & (3)\\ y+u+t=15, & (4)\\ 3x-y+3t-u=49, & (5)\end{cases}$$

Eliminate t and u first according to practical suggestion (**179, 2.**)

Adding (4) and (5), $3x+4t=64;$ (6)
multiplying (3) by 4, $12u+4t=52;$ (7)
subtracting (7) from (6), $3x-12u=12,$
or, $x-4u=4;$ (8)
subtracting (3) from (4), $y-2u=2,$
or, $2y-4u=4;$ (9)
subtracting (9) from (8), $x-2y=0$, or $x=2y;$ (10)
from (1) $6x+3y-6z=120;$ (11)
from (2), $8y-2x+6z=70;$ (12)
by addition, $4x+11y=190;$ (13)
from (10) and (13), $8y+11y=190$, or $y=10;$
by substitution in (10), $x=20;$
" " " (1), $z=5;$
" " " (9), $u=4;$
" " " (3), $t=1.$

11. Given,
$$\begin{cases} x+y-z=1, & (1)\\ 8x+3y-6z=1, & (2)\\ 3z-4x-y=1. & (3)\end{cases}$$

Adding (1) and (3), $2z-3x=2;$ (4)
multiplying (1) by 3, $3x+3y-3z=3;$ (5)
subtracting (2) from (5), $3z-5x=2;$ (6)
" (4) " (6), $z-2x=0$, or $z=2x;$ (7)
substituting value of z in (4,) $4x-3x=2,\ x=2;$
whence, $z=4;$
substituting values of x and z in (1), $y=3.$

12. Given,
$$\begin{cases} 2u+2x+2y+\ \ z=-3, & (1) \\ 3u+3x+3z+2y=\ \ 3, & (2) \\ 4u+4y+4z+3x=-2, & (3) \\ 5x+5y+5z+4u=\ \ 2. & (4) \end{cases}$$

Assuming $u+x+y+z=s$, and adding z to both sides of (1), y to (2), x to (3), and u to (4),

equation (1) becomes $2s=-3+z$, (5)

" (2) " $3s=\ \ 3+y$, (6)

" (3) " $4s=-2+x$, (7)

" (4) " $5s=\ \ 2+u$; (8)

by addition, $14s=s$. (9)

But (9) can be true only when $s=0$; hence by substituting this value of s in (5), (6), (7), and (8), we have

$$z=3,\ y=-3,$$
$$x=2,\ u=-2.$$

13. Given,
$$\begin{cases} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62, & (1) \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47, & (2) \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38. & (3) \end{cases}$$

Clearing (1) of fractions, $6x+4y+3z=\ 744$; (4)

" (2) " $20x+15y+12z=2820$; (5)

" (3) " $15x+12y+10z=2280$; (6)

multiplying (4) by 4, $24x+16y+12z=2976$; (7)

subtracting (5) from (7), $4x+y=\ 156$; (8)

multiplying (4) by 10, $60x+40y+30z=7440$; (9)

" (6) " 3, $45x+36y+30z=6840$; (10)

by subtraction, $15x+4y=\ 600$; (11)

multiplying (8) by 4, $16x+4y=\ 624$;

whence, $x=24$, $y=60$, and $z=120$, *Ans.*

14. Given,
$$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=2, & (1) \\[2ex] \dfrac{1}{x}+\dfrac{1}{z}=3, & (2) \\[2ex] \dfrac{1}{y}+\dfrac{1}{z}=3\,; & (3) \end{cases}$$

adding the three equations, $$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=8\,; \qquad (4)$$

dividing by 2, $$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=4\,, \qquad (5)$$

subtracting (1) from (5), $$\frac{1}{z}=2, \text{ or } z=\tfrac{1}{2}\,;$$

" (2) " (5), $$\frac{1}{y}=1, \text{ or } y=1\,;$$

" (3) " (5), $$\frac{1}{x}=1, \text{ or } x=1.$$

15. Given,
$$\begin{cases} x+a=y+z, & (1) \\ y+a=2x+2z, & (2) \\ z+a=3x+3y\,; & (3) \end{cases}$$

$$\begin{array}{lrr}
\text{by transposition,} & x-y-z=-a\,; & (4) \\
 & -2x+y-2z=-a\,; & (5) \\
 & -3x-3y+z=-a\,; & (6) \\
\text{adding (4) and (6),} & -2x-4y=-2a\,; & (7) \\
\text{or,} & -x-2y=-a\,; & (8) \\
\text{multiplying (4) by 2,} & 2x-2y-2z=-2a\,; & (9) \\
\text{subtracting (5) from (9),} & 4x-3y=-a\,; & (10) \\
\text{multiplying (8) by 4,} & -4x-8y=-4a\,; & (11) \\
\text{adding (10) and (11),} & 11y=\ \ 5a\,; & \\
\text{whence,} & y=\tfrac{5}{11}a,\ x=\tfrac{1}{11}a,\ z=\tfrac{7}{11}a,\ \textit{Ans.} &
\end{array}$$

16. Assume $x+y+z=s$; then the equations become,

$$\begin{array}{lr}
s+z=2(b+c), & (1) \\
s+y=2(a+c), & (2) \\
s+x=2(a+b). & (3)
\end{array}$$

Adding and uniting, $4s=4(a+b+c)$;

$s=a+b+c$;

by substitution in (3), $x=a+b-c$

" " " (2), $y=a+c-b$

" " " (1), $z=b+c-a$ } *Ans.*

17. Given,
$$\begin{cases} 7x-3y=a, & (1) \\ 5y-11x=a, & (2) \\ 9y-5z=a; & (3) \end{cases}$$

multiplying (1) by 5, $35x-15y=5a$; (4)

" (2) by 3, $-33x+15y=3a$; (5)

adding (4) and (5), $2x=8a$;

whence, $x=4a$, $y=9a$, and $z=16a$.

18. Given,
$$\begin{cases} \frac{x}{a}+\frac{y}{b}=\frac{x}{b}-\frac{y}{a}, & (1) \\ x+y=\frac{4^2b}{a^2-b^2}. & (2) \end{cases}$$

Clearing (1) of fractions, $bx+ay=ax-by$;

or, $(a-b)x-(a+b)y=0$; (3)

multiplying (2) by $(a+b)$, $(a+b)x+(a+b)y=\frac{4a^2b}{a-b}$; (4)

adding (3) and (4), $2ax=\frac{4a^2b}{a-b}$, and $x=\frac{2ab}{a-b}$;

substituting value of x in (3), $2ab-(a+b)y=0$, and $y=\frac{2ab}{a+b}$.

19. Given,
$$\begin{cases} ax+by+cz=ab+ac+bc, & (1) \\ a^2x+b^2y+c^2z=3abc, & (2) \\ \frac{x-c}{bc}+\frac{y-c}{ac}+\frac{y-a}{ac}+\frac{z-a}{ab}=0. & (3) \end{cases}$$

Clearing (3) of fractions, and transposing, } $ax+2by+cz=2ac+ab+bc$; (4)

subtracting (1) from (4), $by=ac$, and $y=\frac{ac}{b}$

substituting value of y in (1), $ax+cz=ab+bc$; (5)

" " " " (2), $a^2x+c^2z=2abc$; (6)

multiplying (5) by c, $\quad acx+c^2z=abc+bc^2$; (7)

subtracting (7) from (6), $\quad a(a-c)x=bc(a-c)$, and $x=\frac{bc}{a}$;

substituting value of x in (5), $\quad bc+cz=ab+bc$, and $z=\frac{ab}{c}$.

20 Given,
$$\begin{cases} cx+y+\ az=2a, & (1) \\ c^2x+y+\ a^2z=2ac, & (2) \\ acx-y+\ acz=a^2+c^2. & (3) \end{cases}$$

Adding (1) and (3), $\quad cx+acx+az+acz=2a+a^2+c^2$; (4)
or, $\quad c(1+a)x+a(1+c)z=2a+a^2+c^2$; (5)
adding (2) and (3), $\quad c(c+a)x+a(c+a)z=c^2+2ac+a^2$,
or, dividing by $(c+a)$, $\quad cx+az=c+a$; (6)
multiplying (6) by $(1+c)$, $\quad c(1+c)x+a(1+c)z=c+a+c^2+ac$; (7)
subtracting (7) from (5), $\quad c(a-c)x=a^2-ac+a-c=(a-c)(a+1)$; (8)
whence, $\quad x=a+1$;

substituting in (6), $\quad z=\frac{c-1}{a}$;

" " (1), $\quad y=a-c$.

21. Given,
$$\begin{cases} a^2x+ay\ +az\ =a, & (1) \\ ax+a^2y+az\ =a^2, & (2) \\ ax+ay\ +a^2z=a^3. & (3) \end{cases}$$

Dividing each equation by a, $\quad ax+y+z=1$; (4)
$x+ay+z=a$; (5)
$x+y+az=a^2$; (6)
subtracting (5) from (4), $\quad (a-1)x+(1-a)y=1-a$,
or $x-y=-1$; (7)
" (6) " (2), $\quad (a-1)x+(a^2-1)y=0$;
or $x+(a+1)y=0$; (8)

" (7) " (8), $\quad (a+2)y=1$; or $y=\frac{1}{a+2}$;

substituting in (7), $\quad x=\frac{1}{a+2}-1=-\frac{a+1}{a+2}$;

" " (6), $\quad -\frac{a+1}{a+2}+\frac{1}{a+2}+az=a^2$;

whence, $\quad z=\frac{(a+1)^2}{a+2}$.

PROBLEMS

PRODUCING EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

(**174**, page 119.)

1. Let $x=$ the first number, and $y=$ the second; then by the conditions of the problem, we have

$$2x+3y=105, \qquad (1)$$
$$3x+2y=\ 95; \qquad (2)$$

whence, by elimination, $x=15$, and $y=25$, *Ans.*

By adding the equations, and taking $\frac{2}{5}$ of their sum from the first equation, we have the value of y; taking the same from (2), we have x.

2. Let $x=$ the first, $y=$ the second, $z=$ the third.

By the conditions,
$$\begin{cases} x+\dfrac{y+z}{2}=120, & (1) \\ y+\dfrac{x+z}{5}=\ 90, & (2) \\ z+x+y=190; & (3) \end{cases}$$

clearing (1) of fractions, $2x+y+z=240;$ (4)
subtracting (3) from (4), $x=\ 50;$
clearing (2) of fractions, $5y+x+z=450;$ (5)
subtracting (3) from (5), $4y=260;$
whence, $y=65$, and $z=75$.

3. Let $x=$ A's share, $y=$ B's, and $z=$ C's.

By the conditions,
$$\begin{cases} x-\frac{4}{7}(y+z)=120=a, & (1) \\ y-\frac{3}{8}(x+z)=120=a, & (2) \\ z-\frac{2}{9}(y+x)=120=a; & (3) \end{cases}$$

clearing of fractions, $7x-4y-4z=7a,$ (4)

$-3x+8y-3z=8a,$ (5)

$-2x-2y+9z=9a;$ (6)

multiplying (4) by (2), $14x-8y-8z=14a;$ (7)

adding (7) and (5), $11x-11z=22a$, or $x-z=2a;$ (8)

multiplying (6) by 2, $-4x-4y+18z=18a;$ (9)

subtracting (4) from (9), $-11x+22z=11a$, or $-x+2z=a;$ (10)

adding (10) and (8), $z=3a=360,$

whence, $x=600$, and $y=480$.

4. Let $x=$ A's daily wages, $y=$ B's, and $z=$ C's.

By the conditions, $\begin{cases} 6(x+y)=40, & x+y=6\frac{2}{3}; \quad (1) \\ 9(x+z)=54, & x+z=6; \quad (2) \\ 15(y+z)=80, & y+z=5\frac{1}{3}. \quad (3) \end{cases}$

One half the sum of the equations is $x+y+z=9;$ (4)

subtracting (3) from (4), $x=3\frac{2}{3};$

" (2) " (4), $y=3;$

" (1) " (4), $z=2\frac{1}{3}.$

5. Let x, y, z and u represent their ages respectively.

By the conditions, $\begin{cases} x+y+z=18, & (1) \\ x+y+u=16, & (2) \\ x+z+u=14, & (3) \\ y+z+u=12. & (4) \end{cases}$

One third the sum of the equations is, $x+y+z+u=20;$ (5)

subtracting (4) from (5), $x=8;$

" (3) " " $y=6;$

" (2) " " $z=4;$

" (1) " " $u=2.$

6. Let $x=$ A's shillings, $y=$ B's, and $z=$ C's; then by the conditions of the problem, after the first game each will have as follows:

$x-y-z=$ A's shillings,

$2y=$ B's "

$2z=$ C's "

After the second game,

$$2x-2y-2z= \text{A's shillings,}$$
$$2y-(x-y-z)-2z=3y-x-z= \text{B's} \quad \text{"}$$
$$4z= \text{C's} \quad \text{"}$$

After the third game,

$$4x-4y-4z=16, \quad (1)$$
$$6y-2x-2z=16, \quad (2)$$
$$7z-x-y=16; \quad (3)$$

Adding the equations, $x+y+z=48$; (4)

" (3) and (4), $8z=64$, and $z=8$;

" $\frac{1}{4}$ of (1) to (4), $2x=52$, and $x=26$;

substituting in (4), $y=14$.

7. The same equations as in Ex. 6th, page 116.

8. Let $x=$ the value of the better horse, and $y=$ the value of the poorer.

By the conditions,
$$\begin{cases} x+15=\frac{4}{3}(y+10), & (1) \\ x+10=\frac{15}{13}(y+15); & (2) \end{cases}$$

adding 5 to both sides of (2), $x+15=\frac{15}{13}(y+15)+5$; (3)

comparing (1) and (3), by Ax. 7, $\frac{4}{3}(y+10)=\frac{15}{13}(y+15)+5$; (4)

clearing of fractions, $52y+520=45y+675+195$; (5)

whence, $7y=350$, and $y=50$, poorer horse;

and by substitution in (1), $x=65$, better horse.

9. Let $x=$ the price of a dozen of sherry,

$y=$ " " " " brandy. Put $a=78$.

By the conditions,
$$\begin{cases} 2x+\ y=3a, & (1) \\ 7x+2y=9a+9; & (2) \end{cases}$$

multiplying (1) by 2, $4x+2y=6a$; (3)

subtracting (3) from (2), $3x=3a+9$,

or, $x=a+3=81$, sherry;

whence, from (1), $y=a-6=72$, brandy.

10. Let $x=$ the time in which A can do the work,

$y=$ " " " B " " ";

then $\frac{1}{x}=$ the part of the work A can do in one day,

$\frac{1}{y}=$ " " " B " " ".

In four days, both working together would do $\frac{1}{4}$ of the work.

Therefore,
$$\begin{cases} \frac{4}{x}+\frac{4}{y}=\frac{1}{4}; & (1) \\ \frac{36}{y}=\frac{3}{4};\ \text{or}\ \frac{12}{y}=\frac{1}{4}; & (2) \end{cases}$$

whence, $y=48$, and $x=24$, *Ans.*

11. Let $x=$ numerator, and $y=$ denominator.

By conditions,
$$\begin{cases} \frac{2x}{y+7}=\frac{2}{3}, & (1) \\ \frac{x+2}{2y}=\frac{3}{5}; & (2) \end{cases}$$

clearing of fractions, and transposing, $6x-2y=\ \ 14,$ (3)

$5x-6y=-10;$ (4)

multiplying (3) by 3, $18x-6y=\ \ 42;$ (5)

subtracting (4) from (5), $13x=\ \ 52,$

whence, $x=\ \ 4,\ y=5.$

Hence the fraction is $\frac{4}{5}$, *Ans.*

12. Let $x=$A's money, and $y=$B's.

By the conditions,
$$\begin{cases} x+\frac{2y}{3}=240=a, & (1) \\ y+\frac{3x}{4}=240=a; & (2) \end{cases}$$

clearing of fractions, $3x+2y=3a,$ (3)

$3x+4y=4a;$ (4)

subtracting (3) from (4), $2y=a,\quad y=120;$

substituting in (3), $3x+a=3a,\ x=160.$

13. Let $x=$ the number of persons, and $y=$ what each paid; then $xy=$ the amount of the bill.

By conditions, $\begin{cases}(x+4)(y-1)=xy; & (1)\\ (x-3)(y+1)=xy, & (2)\end{cases}$

expanding (1), $xy+4y-x-4=xy,$ (3)

" (2), $xy-3y+x-3=xy;$ (4)

dropping xy $4y-x-4=0,$ (5)

$-3y+x-3=0;$ (6)

adding (5) and (6), $y-7=0;$

whence, $y=7,\ x=24,$ *Ans.*

14. Let $x=$ the digit in the place of tens, and $y=$ the digit in the place of units; then $10x+y$ will represent the number.

Now by the conditions, $\begin{cases}10x+y=4x+4y, & (1)\\ 10x+y+27=x+10y; & (2)\end{cases}$

transposing and reducing (1), $2x=y;$ (3)

" " " (2), $y-x=3;$ (4)

adding (3) and (4), $x=3$, whence, $y=6$.

Hence, 36, *Ans.*

15. Let x represent the number of hundreds, y the number of tens, and z the number of units; then $100x+10y+z$ will represent the number.

By the conditions, $\begin{cases}x+y+z=11, & (1)\\ z=2x, & (2)\\ 100x+10y+z+297=x+10y+100z; & (3)\end{cases}$

transposing and reducing (3), $z-x=3;$ (4)

substituting value of z in (4), $2x-x=3;$

whence, $x=3,\ z=6,$ and $y=2$.

Hence, 326, *Ans.*

16. Let x, y, and z represent the parts.

By the conditions, $\begin{cases}x+y+z=90, & (1)\\ 2x+40=3y+20, & (2)\\ 4z+10=2x+40; & (3)\end{cases}$

from (2) $y=\dfrac{2x+20}{3}$;

" (3) $z=\dfrac{2x+30}{4}$;

by substitution in (1), $x+\dfrac{2x+20}{3}+\dfrac{2x+30}{4}=90$,

$x=35$, first part,
$y=30$, second "
$z=25$, third "

ANOTHER METHOD.

By the conditions, twice the first part plus 40, three times the second part plus 20, and four times the third part plus 10, are all equal to the same number, which we may represent by x; then

$\dfrac{x-40}{2}=$ the first part,

$\dfrac{x-20}{3}=$ " second "

and $\dfrac{x-10}{4}=$ " third "

Therefore, $\dfrac{x-40}{2}+\dfrac{x-20}{3}+\dfrac{x-10}{4}=90$, (1)

whence, $x=110$.

$\dfrac{x-40}{2}=35$, $\dfrac{x-20}{3}=30$, and $\dfrac{x-10}{4}=25$, *Ans.*

17. Let $x=$ the part at 5 per cent., and y the part at 4 per cent. Put \$100,000$=a$, 4640$=b$.

By the conditions, $\begin{cases} x+y=a, & (1) \\ \dfrac{5x}{100}+\dfrac{4y}{100}=b; & (2) \end{cases}$

from (1), $y=a-x$;

" (2), by substitution, $\dfrac{5x}{100}+\dfrac{4a-4x}{100}=b$,

whence, $x=100b-4a=\$64000$
and $y=\quad a-\quad x=\$36000$ } *Ans.*

18. To avoid the labor of using large numbers in the operation, put $a=5000$; then $2a=10000$, $3a=15000$, $\frac{3a}{10}=1500$, and $\frac{16a}{100}=800$.

Let $x=$ the principal of the first person, and $y=$ the rate,
$x+2a=$ " " " second " $y+1=$ " "
and $x+3a=$ " " " third " $y+2=$ " "

By conditions,
$$\begin{cases} \frac{xy}{100}=\frac{xy+2ay+x+2a}{100}-\frac{16a}{100}, & (1) \\ \frac{xy}{100}=\frac{xy+3ay+2x+6a}{100}-\frac{3a}{10}; & (2) \end{cases}$$

clearing of fractions and dropping xy, $\quad 0=2ay+x+2a-16a$; (3)
$0=3ay+2x+6a-30a$; (4)
multiplying (3) by 2, $\quad 0=4ay+2x+4a-32a$; (5)
subtracting (4) from (5), $\quad ay=4a$, and $y=4$;
substituting in (3), $\quad x=6a=\$30000$.

Hence, $\begin{cases} \text{Principals, \$30,000, \$40,000, \$45,000;} \\ \text{Rates, 4, 5, 6, per cent.} \end{cases}$

19. Let x, y and z represent their respective ages.

By the conditions,
$$\begin{cases} x-y=z, & (1) \\ 5y+2z-x=147, & (2) \\ x+y+z=96; & (3) \end{cases}$$

adding (1) and (3), $\quad 2x=96$, and $x=48$;
substituting in (2), $\quad 5y+2z=195$; (4)
" (3), $\quad y+z=48$; (5)
subtracting twice (5) from (4), $\quad 3y=99$;
whence, $\quad y=33$, $z=15$.

20. Let $x=$ what A is worth, $y=$ what B is worth, and $z=$ what C is worth; also let $s=$ what they all are worth. Put $100=a$.

By the conditions,
$$\begin{cases} x+3y+3z=47a, & (1) \\ y+4x+4z=58a, & (2) \\ z+5x+5y=63a; & (3) \end{cases}$$

adding $2x$ to (1), $\quad 3s=47a+2x$; (4)
" $3y$ to (2), $\quad 4s=58a+3y$; (5)
" $4z$ to (3), $\quad 5s=63a+4z$; (6)

multiplying (4) by 6,	$18s=282a+12x$;	(7)
" (5) by 4,	$16s=232a+12y$;	(8)
" (6) by 3,	$15s=189a+12z$;	(9)
by addition,	$49s=703a+12s$;	
whence,	$s=19a$;	
substituting value of s in (4),	$2x=10a,\ x=5a=500$;	
" " " (5),	$3y=18a,\ y=6a=600$;	
" " " (6),	$4z=32a,\ z=8a=800.$	

21. Let $x=$ the cost of a pound of tea, $y=$ the cost of a pound of coffee; then $50x=$ the whole cost of the tea, and $30y=$ the whole cost of the coffee. His gain in selling the tea is $\frac{1}{10}$ of its cost, or $5x$; his gain in selling the coffee is $\frac{1}{5}$ of its cost, or $6y$.

By the conditions, $\begin{cases} 5x+6y=2.90, & (1) \\ 55x+36y=27.40; & (2) \end{cases}$

subtracting 6 times (1) from (2), $25x=10.00$;

whence, $x=\$.40$, and $y=\$.15$, *Ans.*

22. Let x, y, z, u and v respectively represent their money.

Then by the conditions,
$$\begin{cases} x+\frac{y}{2}=30, & (1) \\ \frac{y}{2}+\frac{z}{3}=30, & (2) \\ \frac{2z}{3}+\frac{u}{4}=30, & (3) \\ \frac{3u}{4}+\frac{v}{6}=30, & (4) \\ \frac{5v}{6}=30, \text{ whence } v=36; & (5) \end{cases}$$

substituting value of v in (4),	$\frac{3u}{4}=24$,	"	$u=32$;
" " u in (3),	$\frac{2z}{3}=22$,	"	$z=33$;
" " z in (2),	$\frac{y}{2}=19$,	"	$y=38$;
" " y in (1),	$x+19=30$,	"	$x=11.$

23. Let x, y and z represent respectively their money. Put $a=1000$.

By the conditions,
$$\begin{cases} x+\frac{y}{2}=2000=2a, & (1) \\ y+\frac{z}{3}=2000=2a, & (2) \\ z+\frac{x}{4}=2000=2a; & (3) \end{cases}$$

clearing of fractions,
$$\begin{cases} 2x+y=\ 4a, & (4) \\ 3y+z=\ 6a, & (5) \\ 4z+x=\ 8a; & (6) \end{cases}$$

multiplying (5) by 4, $\quad 12y+4z=24a;\quad$ (7)

subtracting (6) from (7), $\quad 12y-x=16a;\quad$ (8)

" (8) from 12 times (4), $\quad 25x=32a;$

whence, $\quad x=\frac{32a}{25};$

or, $\quad x=1280;$

by substitution, $\quad y=1440;$

and $\quad z=1680.$

24. Let $x=$ the hourly rate of the first courier, and $y=$ the hourly rate of the second courier. Now the distance divided by the rate will be the time; hence,

$$\frac{147}{x}-\frac{147}{y}=28; \qquad (1)$$

$$\frac{17}{x}+\frac{56}{y}=13\tfrac{2}{3}. \qquad (2)$$

These equations are the same as those in Ex. 19, page 111, and may be solved in a similar manner.

25. Let $x=$ the greater, and $y=$ the less.

By conditions,
$$\begin{cases} \frac{x}{2}+\frac{y}{3}=13, & (1) \\ \frac{x}{3}-\frac{y}{2}=\ 0; & (2) \end{cases}$$

multiplying (1) by 3, $\frac{3x}{2}+y=39$; (3)

" (2) by 2, $\frac{2x}{3}-y=\ 0$; (4)

by addition, $\frac{13x}{6}=39$;

whence, $x=18$, and $y=12$, *Ans.*

26. Let $x=$ the first, $y=$ the second, and $z=$ third.

By conditions,
$$\begin{cases} x+\frac{1}{2}(y+z)=51=a, & (1)\\ y+\frac{1}{3}(x+z)=51=a, & (2)\\ z+\frac{1}{4}(x+y)=51=a; & (3)\end{cases}$$

clearing of fractions,
$$\begin{cases} x+(x+y+z)=2a, & (4)\\ 2y+(x+y+z)=3a, & (5)\\ 3z+(x+y+z)=4a. & (6)\end{cases}$$

Let $(x+y+z)=s$, and multiply (4) by 6, (5) by 3, and (6) by 2;

equation (4) becomes $6x+6s=12a$; (7)

" (5) " $6y+3s=\ 9a$; (8)

" (6) " $6z+2s=\ 8a$; (9)

by addition, $6s+11s=29a$, or $s=87$.

substituting value of s in (4), $x=2a-s$, or $x=15$;

" " " (5), $2y=3a-s$, or $y=33$;

" " " (6), $3z=4a-s$, or $z=39$.

27. Let $x=$ A's, $y=$ B's, and $z=$ C's sheep.

By the conditions,
$$\begin{cases} x+8-4=y+z-8, & (1)\\ \frac{1}{2}(y+8)=x+z-8, & (2)\\ \frac{1}{3}(z+8)=x+y-8; & (3)\end{cases}$$

clearing of fractions and uniting, $x-y-z=-12$, (4)

$-2x+y-2z=-24$, (5)

$-3x-3y+z=-32$; (6)

adding (4) and (6), $-2x-4y=-44$; (7)

or $x+2y=22$; (8)

subtracting twice (4) from (5), $-4x+3y=0$; (9)

adding four times (8) to (9), $11y=88$;

whence, $y=\ 8$, $x=\ 6$, and $z=10$.

28. Let $\frac{x}{y}$ represent the fractions.

By the conditions,
$$\begin{cases} \dfrac{x+1}{y}=\dfrac{1}{3}, & (1) \\ \dfrac{x}{y+1}=\dfrac{1}{4}; & (2) \end{cases}$$

clearing of fractions, &c., $3x-y=-3$; (3)

$4x-y=\ \ 1$; (4)

whence, by subtraction, $x=\ \ 4$, and $y=15$;

consequently the fraction is $\frac{4}{15}$, *Ans.*

29. Let $\frac{x}{y}$ represent the fraction.

By the conditions,
$$\begin{cases} \dfrac{x+2}{y}=\dfrac{5}{7}, & (1) \\ \dfrac{x}{y+2}=\dfrac{1}{3}; & (2) \end{cases}$$

clearing of fractions, &c., $7x-5y=-14$; (3)

$3x-y=\ \ 2$; (4)

subtracting (3) from five times (4), $8x=\ \ 24$, whence $x=3$; (5)

substituting in (4), $y=7$;

hence the fraction is, $\frac{3}{7}$, *Ans.*

30. Let x, y, z, and u represent the parts of an acre which the men respectively can mow in one hour.

By the conditions,
$$\begin{cases} x+3y+\ 2z+\ 2u=1, & (1) \\ 3x+2y+\ 4z+11u=2, & (2) \\ 5x+4y+12z+\ 5u=3, & (3) \\ 9x+7y+\ 6z+\ 8u=4, & (4) \end{cases}$$

subtracting twice (1) from (2), $x-\ 4y+\ 7u=\ \ 0$; (5)

" six times (1) from (3), $-x-14y-\ 7u=-3$; (6)

adding (5) and (6), $18y=3$, whence $y=\frac{1}{6}$;

subtracting (4) from three times (1), $-6x+2y-2u=-1$; (7)

substituting the value of y in (5) and (7), $x-\frac{2}{3}+7u=\ \ 0$; (8)

$-6x+\frac{1}{3}-2u=-1$; (9)

transposing, and multiplying (8) by 6, $\qquad 6x+42u=4, \qquad (10)$

adding (10) and (9), $\qquad 40u=\frac{8}{3},\ u=\frac{1}{15},$

by substitution, $\qquad x=\frac{1}{5}$, and $z=\frac{1}{12}$.

Hence, A can mow an acre in 5 hours; B in 6 hours; C in 12 hours; and D in 15 hours.

31. Let $x=$ A's money, and $y=$B's.

By the conditions,
$$\begin{cases} x-5=\frac{1}{3}(y+5), & (1) \\ x+5=3(y-5); & (2) \end{cases}$$

by subtraction, $$10=3y-15-\frac{y}{2}-\frac{5}{2}; \qquad (3)$$

whence, $\qquad 20=5y-35$, and $y=11$;

by substitution, $\qquad x=13$.

32. Let $\qquad x=$ the number of bushels of wheat flour,

and $\qquad y=$ " " " " barley "

Then $\quad 10x+4y=$ the cost of the whole,

and $\quad 11x+11y=$ the value of the sale at 11 shillings per bushel.

Now, $\qquad \frac{143\frac{3}{4}}{100}=\frac{575}{400}$; and by the conditions,

$$\tfrac{575}{400}(10x+4y)=11x+11y, \qquad (1)$$

or, $$5750x+2300y=4400x+4400y, \qquad (2)$$

or, $$1350x=2100y,$$

or, $$9x=14y.$$

Converting the equation into a proportion, by placing the factors of one member for the extremes, and the factors of the other for the means, we have

$$x : y :: 14 : 9$$

or, $\qquad$ wheat : barley :: 14 : 9, *Ans.*

33. Let x represent the number of tens, and y the number of units; then the number expressed in units, must be $10x+y$.

By the conditions,
$$\begin{cases} \dfrac{10x+y}{5}=2x+\dfrac{1}{5}, & (1) \\[2ex] \dfrac{10x+y}{8}=5y+\dfrac{1}{8}=\dfrac{41}{8}. & (2) \end{cases}$$

Reducing (1), we have $y=1$,
" (2), " " $x=4$.

The number, therefore, is 41, *Ans.*

34. Let $x=$ the seniors, $y=$ the juniors, $z=$ the sophomores, and $u=$ the freshmen. Place $119=a$.

Now by the conditions of the problem, we have

$$x+\frac{y}{2}=a, \quad (1) \qquad y+\frac{z}{3}=a, \quad (2)$$

$$z+\frac{u}{4}=a, \quad (3) \qquad u+\frac{x}{5}=a; \quad (4)$$

subtracting (2) from twice (1), $2x-\frac{z}{3}=a$; (5)

adding (3) to three times (5), $6x+\frac{u}{4}=4a$; (6)

subtracting (4) from four times (6), $24x-\frac{x}{5}=15a$;

or, $119x=75a$;
whence, $x=75$, $y=88$, $z=93$, and $u=104$.

35. Let x, y, z, and u represent the numbers respectively.

By the conditions,
$$\begin{cases} 3x+y=a=359, & (1) \\ 4y+z=a, & (2) \\ 5z+u=a, & (3) \\ 6u+x=a; & (4) \end{cases}$$

subtracting (2) from four times (1), $12x-z=3a$; (5)
adding (3) to five times (5), $60x+u=16a$; (6)
subtracting (4) from six times (6), $359x=95a$;
whence,

$$x=95,\ y=74,\ z=63, \text{ and } u=44.$$

GENERAL SOLUTION OF PROBLEMS.

(179, page 126.)

1. Let $x=$ the greater part, and $y=$ the less.

By conditions, $\begin{cases} x+y=n, & (1) \\ x+a=y+b; & (2) \end{cases}$

transposing (2), $x-y=b-a;$ (3)

adding (1) and (3), $2x=n+b-a, \quad x=\dfrac{n+b-a}{2};$

subtracting (3) from (1), $2y=n+a-b, \quad y=\dfrac{n+a-b}{2}.$

2. Substituting the numerals, $\begin{cases} x=\dfrac{84+58-16}{2}=63; \\ y=\dfrac{84+16-58}{2}=21. \end{cases}$

3. Let x, y, and z represent the numbers.

By the conditions, $\begin{cases} x+y+z=s, & (1) \\ x=y-a, & (2) \\ y=z-b, & (3) \end{cases}$

subtracting (3) from (2), $x-2y+z=b-a;$ (4)

" (4) " (1), $3y=s+a-b$, or $y=\dfrac{s+a-b}{3};$

substituting in (2), $x=\dfrac{s+a-b}{3}-a$, or $x=\dfrac{s-2a-b}{3};$

" " (3) $z=\dfrac{s+a-b}{3}+b$, or $z=\dfrac{s+a+2b}{3}.$

4. Let $x=$ my indebtedness to A;
then $nx=$ " " B;
and $mx=$ " " C.
Now by the conditions, $x+nx+mx=a;$

whence, $x=\dfrac{a}{1+n+m}$, *Ans.*

5. Substituting the numerals, $x=\frac{\$786}{1+2+3}=\$131$, *Ans.*

6. Let $x=$ the number of days he was idle, and

$a-x=$ " " " worked;

then $(a-x)b=$ the money due for his labor, and

$cx=$ " " he forfeited.

Now by the conditions, we have

$$(a-x)b-cx=d;$$

whence, $x=\frac{ab-d}{b+c}$, *Ans.*

ANOTHER METHOD.

6. Let $x=$ the number of days he was idle. He would have received ab cents, had he worked all the time. Every day he was idle he lost his wages, b, and forfeited c cents; his whole loss, therefore, for the idle days, was $(b+c)x$. Therefore,

$$ab-(b+c)x=d;$$

whence, $x=\frac{ab-d}{b+c}$, *Ans.*

7. Let $x=$ the value of the saddle, and $nx=$ the value of the horse; then

$$(n+1)x=a;$$

whence, $x=\frac{a}{n+1}$, and $nx=\frac{na}{n+1}$, *Ans.*

8. Let $x=$ the rent last year; then by the conditions,

$$x+\frac{nx}{100}=a;$$

whence, $x=\frac{100a}{100+n}$, *Ans.*

9. Let x represent his income; then by the conditions,

$$\frac{x}{7}+a+\frac{x}{3}+b=x;$$

$$11x=21(a+b),$$

whence,
$$x=\frac{21(a+b)}{11}, \textit{ Ans.}$$

10. Let x represent his income; then by conditions,

$$\frac{x}{m}+a+\frac{x}{n}+b=x;$$

whence,
$$nx+mx+mn(a+b=mnx),$$

$$x(mn-m-n)=mn(a+b),$$

$$x=\frac{mn(a+b)}{mn-m-n}, \textit{ Ans.}$$

11. Let $x=$ the number of days required when all work together. Since A can do the work in a days, in one day he will do a part of the whole work expressed by $\frac{1}{a}$, and in x days he will do a part of the work expressed by $\frac{x}{a}$. Reasoning similarly, in x days B will do a part of the work expressed by $\frac{x}{b}$, and C a part of the work expressed by $\frac{x}{c}$. Therefore,

$$\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=1;$$

whence,
$$x(ab+ac+bc)=abc,$$

$$x=\frac{abc}{ab+ac+bc}, \textit{ Ans.}$$

12. Substituting the numerals, $x=\frac{6\times 8\times 12}{48+72+96}=\frac{576}{216}=2\frac{2}{3}$, *Ans.*

13. Let x represent the number; then by the conditions,

$$ax-c=bx+d;$$

whence,
$$(a-b)x=c+d,$$

$$x=\frac{c+d}{a-b}, \textit{ Ans.}$$

14. Let $x=$ the number of bushels of oats,

$y=$ " " " peas.

Then by the conditions, $\begin{cases} x+y=c, & (1) \\ ax+by=cd\,; & (2) \end{cases}$

subtracting b times (1) from (2), $(a-b)x=c(d-b)\,;$ (3)

whence, $x=\dfrac{c(d-b)}{a-b}$, oats;

subtracting (2) from a times (1), $(a-b)y=c(a-d)\,;$

whence, $y=\dfrac{c(a-d)}{a-b}$, peas

15. Let $x=$ the number of nuts. The party of a boys lost bm nuts by the snatching, and the party of b boys lost am nuts. The party of a boys had $x-bm$ nuts to be divided, and each would have $\dfrac{x-bm}{a}$. The party of b boys had $x-am$ nuts to be divided, and each would have $\dfrac{x-am}{b}$. Therefore,

$$\frac{x-bm}{a}=\frac{x-am}{b}\,;$$

whence, $(b-a)x=m(b^2-a^2)$,

$$x=\frac{m(b^2-a^2)}{b-a}=m(a+b),\ \textit{Ans.}$$

16. Let x, y, z, and u respectively represent the numbers.

By the conditions, $\begin{cases} ax+y=m, & (1) \\ by+z=m, & (2) \\ cz+u=m, & (3) \\ du+x=m. & (4) \end{cases}$

Subtracting (2) from b times (1), $abx-z=m(b-1)\,;$ (5)

adding (3) to c times (5), $abcx+u=m(bc-c+1)\,;$ (6)

subtracting (4) from d times (6), $abcdx-x=m(bcd-cd+d-1)\,;$ (7)

whence, $x=\dfrac{m(bcd-cd+d-1)}{abcd-1}\,;$

substituting in (4), $$du = m - \frac{m(bcd - cd + d - 1)}{abcd - 1},$$

or, $$du = \frac{m(abcd - bcd + cd - d)}{abcd - 1},$$

or, $$u = \frac{m(abc - bc + c - 1)}{abcd - 1}; \qquad (8)$$

substituting value of u in (3), $$cz = m - \frac{m(abc - bc + c - 1)}{abcd - 1},$$

or, $$cz = \frac{m(abcd - abc + bc - c)}{abcd - 1};$$

dividing by c, $$z = \frac{m(abd - ab + b - 1)}{abcd - 1};$$

substituting in (2), $$y = \frac{m(acd - ad + a - 1)}{abcd - 1}.$$

17. Let $x =$ the number of pupils;
then $ax - c =$ the whole attendance in one school,
and $bx - d =$ " " " the other school.
$na =$ the number of days' attendance for which A pays,
$mb =$ " " " " B "

Now either patron will pay such a part of the whole money raised in his school, as his number of days' attendance is part of the whole attendance. Therefore, by the conditions,

$$\frac{mb}{bx - d} = \frac{na}{ax - c};$$

whence, $$abmx - cbm = anbx - adn;$$

or, $$x(ab)(m - n) = bcm - adn;$$

$$x = \frac{bcm - adn}{ab(m - n)}, \textit{ Ans.}$$

18. Let the several parts be represented by x, ax, a^2x, and a^3x respectively. Then by addition,

$$a^3x + a^2x + ax + x = m;$$

whence, $$x = \frac{m}{a^3 + a^2 + a + 1}, \textit{ Ans.}$$

19. Let x and y represent the numbers.

By the conditions, $\begin{cases} x+y=s, & (1) \\ x-y=d; & (2) \end{cases}$

by addition, $2x=s+d$, and $x=\dfrac{s+d}{2}$;

by subtraction, $2y=s-d$, and $y=\dfrac{s-d}{2}$.

20. Let x, y and z represent the numbers.

By the conditions, $\begin{cases} x+y=a, & (1) \\ x+z=b, & (2) \\ y+z=c; & (3) \end{cases}$

subtracting (3) from the sum of (1) and (2), $2x=a+b-c$,

$$x=\frac{a+b-c}{2};$$

" (2) " " (1) and (3), $2y=a+c-b$,

$$y=\frac{a+c-b}{2};$$

" (1) " " (2) and (3), $2z=b+c-a$,

$$z=\frac{b+c-a}{2}.$$

21. Let $x=$ the number of tens;
and $y=$ " " units;
then $10x+y=$ the number of units in the given number.

By the conditions, $\begin{cases} 10x+y \quad =ax+ay, & (1) \\ 10x+y+c=10y+x; & (2) \end{cases}$

or, $\begin{cases} (10-a)x+(1-a)y=0, & (3) \\ 9x-9y=-c; & (4) \end{cases}$

multiplying (3) by 9, $9(10-a)x+9(1-a)y=0$; (5)

" (4) by $(1-a)$, $9(1-a)x-9(1-a)y=ac-c$; (6)

by addition, $9(11-2a)x=c(a-1)$,

whence, $x=\dfrac{c(a-1)}{9(11-2a)}$, tens;

substituting in (4), $\dfrac{c(a-1)}{11-2a}+c=9y$,

$$y=\frac{c(10-a)}{9(11-2a)}, \text{ units.}$$

INTERPRETATION OF NEGATIVE RESULTS.

(**182**, page 133.)

1. Let x represent the number; then by the conditions,

$$\frac{x}{4}-\frac{x}{3}=12;$$

whence, $x=144$, *Ans.*

In an arithmetical sense, the fourth part of a number can never exceed its third part; and the absurdity of this supposition is indicated by the negative result. Hence the question must be modified so as to read,

What number is that whose third part exceeds its fourth part by 12?

2. Let x equal the number of years; then by the conditions,

$$30+x=3(15+x);$$
$$x=-7\tfrac{1}{2},\ \textit{Ans.}$$

As the man was 30 years old, and his wife 15 years old, at the time of marriage, she is already one half as old as he; he can, therefore, never become three times as old as she. Hence, the problem must be modified as follows:

A man was 30 *years old when he married, and his wife* 15. *How many years before their marriage was his age three times the age of his wife?*

3. By the conditions, $\begin{cases} x+y=s, \\ x-y=d, \end{cases}$ whence, $\begin{cases} x=\dfrac{s+d}{2}, \\ y=\dfrac{s-d}{2}. \end{cases}$

If we make $s=120$ and $d=160$, we have

$$x=\frac{120+160}{2}=140, \text{ greater};$$

$$y=\frac{120-160}{2}=-20, \text{ less.}$$

Algebraically considered, -20 added to 140 is 120, the *sum*; and -20 subtracted from 140 is 160, the *difference.*

4. Let $x=$ B's money; then $3x=$ A's money. Now by the conditions of the problem, we have

$$3x+400=2(x+150);$$

whence,
$$x=-100,$$
$$3x=-300.$$

The negative results show *indebtedness*, instead of capital, at the commencement of business. Hence, the question should read as follows:

A owes three times as much as B. A gains by trading \$400, *and B* \$150, *when A has twice as much money as B. What was the indebtedness of each at first?*

5. Let $x=$ the father's wages, and $y=$ the son's.

By the conditions,
$$\begin{cases} 7x+3y=22, & (1) \\ 5x+\ \ y=18; & (2) \end{cases}$$

subtracting (1) from three times (2), $8x=32$;

or, $x=4$;

whence, $y=-2$.

The negative value of y shows that the boy was charged for board each day two shillings more than his wages.

6. Let x, y and z represent the wages of the man, his wife, and son, respectively.

By the conditions,
$$\begin{cases} 10x+\ 8y+\ 6z=10.30, & (1) \\ 12x+10y+\ 4z=13.20, & (2) \\ 15x+10y+12z=13.85; & (3) \end{cases}$$

subtracting (3) from twice (1), $5x+6y=6.75$; (4)

" (3) from three times (2), $21x+20y=25.75$; (5)

multiplying (4) by 10, $50x+60y=67.50$; (6)

" (5) by 3, $63x+60y=77.25$; (7)

subtracting (6) from (7), $13x=9.75$;

whence, $x=.75$, $y=.50$, and $z=-.20$.

The negative value of z shows that the boy was charged for board \$.20 each day above his daily wages.

7. Let x, y and z represent the wages of each respectively.

By the conditions,
$$\begin{cases} 10x+4y+3z=11.50, & (1) \\ 9x+8y+6z=12.00, & (2) \\ 7x+6y+4z=\ \ 9.00; & (3) \end{cases}$$

subtracting (2) from twice (1), $\quad 11x=11.00;$

whence, $\quad x=1.00,\ y=0,$ and $z=\ .50.$

The value of y shows that the wife's board and wages were equal.

8. Let $x=$ the numerator, and $y=$ the denominator.

By the conditions,
$$\begin{cases} \dfrac{x+1}{y}=\dfrac{3}{5}, & (1) \\[2ex] \dfrac{y}{y+1}=\dfrac{5}{7}. & (2) \end{cases}$$

From (1) $\quad 5x-3y=-5;$

" (2) $\quad 7x-5y=\ \ 5;$

$$25x-15y=-25,$$
$$21x-15y=\ \ 15,$$
$$4x=-40,$$

$$x=-10,\ y=-15;\ \text{hence, } \frac{-10}{-15},\ \textit{Ans.}$$

The modified example will be as follows:

What fraction is that which becomes $\frac{3}{5}$ when 1 is subtracted from its numerator, and $\frac{5}{7}$ when 1 is subtracted from its denominator?

The equations will now be,

$$\frac{x-1}{y}=\frac{3}{5},\quad \frac{x}{y-1}=\frac{5}{7};$$

whence, $\quad x=10,\ y=15;$ hence $\frac{10}{15}$, *Ans.*

9. Let x, y, z and u represent respectively the net capital or insolvency of each.

By the conditions,
$$\begin{cases} x+y+z+u=5780, & (1) \\ x+y+z\qquad\ =7950, & (2) \\ \qquad y+z+u=2220, & (3) \\ x\qquad +z+u=7320. & (4) \end{cases}$$

Subtracting (3) from (1), $x = 3560$, A's net capital.
" (4) " (1), $y = -1540$, B's net insolvency,
" (2) " (1), $u = -2170$, D's net insolvency,
by substitution, $z = 5930$, C's net capital.

The positive values of x and z show net *capital* for A and C; the negative values of y and u show net *insolvency* for B and D.

10. Let $x =$ the number of hours after six o'clock, when A passed B. Now, at six o'clock, B's distance from Boston was $n+4b$ miles. Hence, at the moment of passing we have

$$m-ax = \text{A's distance from Boston};$$
$$n+4b-bx = \text{B's} \quad " \quad " \quad " .$$

Therefore,
$$m-ax=n+4b-bx;$$
$$x=\frac{m-n-4b}{a-b} \text{ hours, } \textit{Ans.}$$

11. Substituting the given values for m, n, a, and b, we have

$$x=\frac{36-28-12}{5-3}=\frac{-4}{2}=-2.$$

That is, A passed B -2 hours *after* six, or 2 hours *before* six, which is 4 o'clock.

12. Let $x =$ the greater, and $x-a =$ the less; then by the conditions,

$$x+5(x-a)=b;$$

whence,
$$x=\frac{b+5a}{8}, \text{ the greater};$$

and
$$x-a=\frac{b-3a}{8}, \text{ the less.}$$

If $a=24$ and $b=48$, then

$$x=\frac{48+120}{8}=21, \text{ greater}; \quad x-a=\frac{48-120}{8}=-9, \text{ less.}$$

Arithmetically speaking, there is no number -9. But considering the quantities, 21 and -9, *algebraically*, they fulfil the conditions of the problem.

(**193**, page 142.)

1. Let x, y, z, and u represent the parts of the contents of the cistern which will flow through the pipes respectively in one hour.

By the conditions,
$$\begin{cases} 15x+15y+15z+15u=1, & (1) \\ 5x+\ 8y+\ 7z+\ 3u=\frac{1}{2}, & (2) \\ 3x+\ 4y+\ 3z+\ \ u=\frac{1}{3}, & (3) \\ 4x+\ 2y+\ 3z+\ 2u=\frac{1}{4}. & (4) \end{cases}$$

Subtracting $\frac{1}{5}$ of (1) from (2) $\quad 2x+5y+4z=\frac{3}{10}, \qquad (5)$

" $\frac{1}{15}$ of (1) " (3), $\quad 2x+3y+2z=\frac{4}{15}, \qquad (6)$

" $\frac{2}{15}$ of (1) " (4), $\quad 2x \quad + z=\frac{7}{60}; \qquad (7)$

multiplying (5) by 3, $\quad 6x+45y+12z=\frac{9}{10}; \qquad (8)$

" (6) by 5, $\quad 10x+15y+10z=\frac{20}{15}; \qquad (9)$

subtracting (8) from (9), $\quad 4x \quad -2z=\frac{13}{30}; \qquad (10)$

multiplying (7) by 2, $\quad 4x \quad +2z=\frac{14}{60}; \qquad (11)$

adding (10) and (11), $\quad 8x=\ \frac{40}{60}$, and $x=\ \frac{5}{60}=\ \frac{1}{12}$;

subtracting (10) and (11) $\quad 4z=-\frac{12}{60}$, and $z=-\frac{3}{60}=-\frac{1}{20}$;

substituting in (5) $\quad 5y=\ \frac{20}{60}$, and $y=\ \frac{4}{60}=\ \frac{1}{15}$;

" (1) $\quad 15u=-\frac{30}{60}$, and $u=-\frac{2}{60}=-\frac{1}{30}$.

Therefore, the contents of the cistern will flow through the first pipe in 12 hours, through the second in 15 hours, through the third in 20 hours, and through the fourth in 30 hours. The positive values of x and y indicate receiving pipes; the negative values of z and u indicate discharging pipes.

2. Let m represent what A will have in 2 days; then $m+2$ will be what B will have in 4 days.

Let $x=$ the number of days hence, when A and B have the same money; then A will have $m+5(x-2)$ dollars, and B will have $m+2+3(x-4)$ dollars. Hence,

$$m+5(x-2)=m+2+3(x-4);$$
$$(5-3)x=0;$$
$$x=\frac{0}{5-3}=0.$$

That is, they now have the same sum.

3. Let x represent the period; then by the conditions,

$$3x-10=\tfrac{3}{4}(4x+8);$$

whence,

$$12x-40=12x+24;$$

$$(12-12)x=64;$$

$$x=\frac{64}{12-12}=\frac{64}{0}=\infty.$$

The value of x is an expression for infinity, according to (**188**, 1.). The period of the comet, therefore, is a number of years greater than any assignable number.

4. Let x represent the monthly wages; then by the conditions,

$$2(9x-450)=3(6x-300);$$

whence,

$$18x-900=18x-900;$$

$$(18-18)x=900-900,$$

$$x=\frac{900-900}{18-18}=\frac{0}{0}.$$

The value of x is a symbol of indetermination, according to (**188**, 4.) The monthly wages of each may therefore be any number of dollars. If they receive more than \$50 a month, they will each lay up the same sum. If they receive less than \$50, they will become equally indebted.

INEQUALITIES.

(**201**, Page 149.)

1. $$5x > \frac{3x}{2}+14;$$

clearing of fractions, $10x > 3x+28$;

dropping $3x$, $7x > 28$; whence, $x > 4$, *Ans.*

2. $$\frac{2x}{5}-\frac{2x}{3} > \frac{2x}{5}-2;$$

dropping $\frac{2x}{5}$, $-\frac{2x}{3} > -2$;

changing signs by (**199**, III.), $\frac{2x}{3} < 2$, or $\frac{x}{3} < 1$;

clearing of fractions, $x < 3$, *Ans.*

3. $$\frac{5x}{8}+\frac{5}{4} < \frac{11}{6}+\frac{7x}{12};$$

clearing of fractions, $$75x+150 < 220+70x;$$
whence, $$5x < 70;$$
$$x < 14, \textit{ Ans.}$$

4. $$\frac{3x}{4}-\frac{x-1}{2} < 6x-\frac{20x+13}{4};$$

whence, $$3x-2x+2 < 24x-20x-13;$$
or, $$-3x < -15;$$
changing all the signs by (**199**, III.), $$3x > 15;$$
$$x > 5, \textit{ Ans.}$$

5. $$ax-b > cx+d;$$
transposing, $$ax-cx > b+d;$$
dividing by $(a-c)$, $$x > \frac{b+d}{a-c}, \textit{ Ans.}$$

6. $$\frac{x-a}{b} < 1-\frac{x}{a};$$

multiplying by ab, $$ax-a^2 < ab-bx;$$
transposing, $$ax+bx < a^2+ab;$$
dividing by $(a+b)$, $$x < a, \textit{ Ans.}$$

7. $$(a-x)(m-x)-a(m-c) < x^2-\frac{a^2c}{m};$$

removing parentheses, $$am-ax-mx+x^2-am+ac < x^2-\frac{a^2c}{m};$$

transposing, $$-ax-mx < -ac-\frac{a^2c}{m};$$

changing all the signs, $$ax+mx > ac+\frac{a^2c}{m};$$

multiplying both sides by m, $$m(a+m)x > ac(a+m);$$

dividing by $m(a+m)$, $$x > \frac{ac}{m}, \textit{ Ans.}$$

(**202**, page 150.)

1. Given, $\begin{cases} 2x+4y > 30, & (1) \\ 3x+2y = 31; & (2) \end{cases}$

dividing (1) by 2, $x+2y > 15;$ (3)
subtracting (3) from (2), (**199**, II.), $2x < 16;$
whence, $x < 8.$

If we substitute 8 for x in (2), the first member will be greater than the second member; thus,

$$24+2y > 31;$$

transposing, $2y > 7;$
whence, $y > 3\frac{1}{2}.$

2. Given, $\begin{cases} 4x-3y < 15, & (1) \\ 8x+2y = 46; & (2) \end{cases}$

multiplying (1) by 2, $8x-6y < 30;$ (3)
subtracting (3) from (2), (**199**, II.), $8y > 16;$
whence, $y > 2,$

Substituting 2 for y in (2) will make the first member less than the second; hence,

$$8x+4 < 46;$$

transposing, $8x < 42;$
whence, $x < 5\frac{1}{4}.$

3. Given, $\begin{cases} 7x-10y < 59, & (1) \\ 4x+\ 5y = 68; & (2) \end{cases}$

multiplying (2) by 2, $8x+10y = 136;$ (3)
adding (1) and (3), $15x < 195;$
whence, $x < 13;$
substituting in (2), $52+5y > 68;$
$5y > 16;$
whence, $y > 3\frac{1}{5}.$

4. Given, $\begin{cases} 5x+3y > 121, & (1) \\ 7x+4y = 168; & (2) \end{cases}$

From equation (2), $y=42-\frac{7x}{4};$ (3)

substituting this value of y in (1), we have

$$5x+126-\frac{21x}{4} > 121; \qquad (4)$$

multiplying by 4, $\quad 20x+504-21x > 484;$

transposing and uniting, $\quad -x > -20;$

whence by (**199**, III.), $\quad x < 20.$

Substituting 20 for x in (2), we have

$$140+4y > 168;$$

transposing, $\quad 4y > 28;$

$$y > 7.$$

5. Given,

$$\begin{cases} \dfrac{x-4}{8}-\dfrac{y-10}{6} > 1, & (1) \\[2ex] \dfrac{3x-24}{4}+\dfrac{x-y}{2} = 13; & (2) \end{cases}$$

multiplying (1) by 24, $\quad 3x-12-4y+40 > 24; \qquad (3)$

or, $\quad 3x-4y > -4; \qquad (4)$

multiplying (2) by 4, $\quad 3x-24+2x-2y = 52; \qquad (5)$

or, $\quad 5x-2y = 76; \qquad (6)$

multiplying (6) by 2, $\quad 10x-4y = 152; \qquad (7)$

subtracting (4) from (7), $\quad 7x < 156;$

whence, $\quad x < 22\frac{2}{7},$

substituting in (6), $\quad 111\frac{3}{7}-2y > 76,$

transposing, $\quad -2y > -35\frac{3}{7}$

or, $\quad 2y < 35\frac{3}{7}$

$$y < 17\frac{5}{7}.$$

INVOLUTION.

(**216**, page 157.)

1. By simple multiplication according to the rule, we shall have

$$(2x^2+3y)^2 = 4x^4 + 6x^2y + 6x^2y + 9y^4$$
$$= 4x^3 + 12x^2y + 9y^4, \textit{ Ans.}$$

2. Multiplying the factor $5x-y^2$ by itself, we get

$$(5x^2-y^2)^2=25x^2-10xy^2+y^4.$$

Multiplying this result by $5x-y^2$,

$$\begin{aligned}(5x-y^2)^3&=125x^3-50x^2y^2+5xy^4-25x^2y^2+10xy^4-y^6\\&=125x^3-75x^2y^2+15xy^4-y^6,\ \textit{Ans.}\end{aligned}$$

3. $$\begin{aligned}(1+2x-3x^2)^2&=1+4x-6x^2+4x^2-12x^3+9x^4\\&=1+4x-2x^2-12x^3+9x^4,\ \textit{Ans.}\end{aligned}$$

4. $$(3a+2b+c)^2=9a^2+12ab+6ac+4b^2+4bc+c^2.$$

Multiplying this result by $3a+2b+c$, we have

$$\begin{aligned}&(3a+2b+c)^3\\&\quad=27a^3+36a^2b+18a^2c+12ab^2+12abc+3ac^2\\&\qquad+18a^2b+(8b^3)+24ab^2+12abc+(8b^2c)+2bc^2\\&\qquad+9a^2c+(4b^2c)+12abc+6ac^2+4bc^2+c^3\\&\quad=27a^3+54a^2b+27a^2c+36ab^2+36abc+8b^3+9ac^2+12b^2c\\&\qquad+6bc^2+c^3,\ \textit{Ans.}\end{aligned}$$

In Examples 5th and 6th we have the simple binomial; see (**70**).

7. $$\begin{aligned}(a^2c^{-2}+a^{-2}c^2)^2&=a^4c^{-4}+2a^0c^0+a^{-4}c^4\\&=a^4c^{-4}+2+a^{-4}c^4,\ \textit{Ans.}\end{aligned}$$

8. $$\begin{aligned}(a^2+1+a^{-2})^2&=a^4+2a^2+2a^0+1+2a^{-2}+a^{-4}\\&=a^4+2a^2+3+2a^{-2}+a^{-4}.\end{aligned}$$

Multiplying again by a^2+1+a^{-2},

$$\begin{aligned}(a^2+1+a^{-2})^3&=a^6+2a^4+3a^2+2a^0+a^{-2}\\&\qquad+a^4+2a^2+3+2a^{-2}+a^{-4}\\&\qquad+a^2+2a^0+3a^{-2}+2a^{-4}+a^{-6}\\(a^2+1+a^{-2})^3&=a^6+3a^4+6a^2+7+6a^{-2}+3a^{-4}+a^{-6},\ \textit{Ans.}\end{aligned}$$

For Example 9th, see (**71**).

(**217**, page 159.)

All examples in this article are readily solved by a strict application of the rule, which is an important one. In the first three examples the *symmetry* of the answers should be noticed.

6. $$\begin{array}{rl}(1-a+a^2-a^3)^2 = & 1-2a+2a^2-2a^3 \\ & \qquad +\ a^2-2a^3+2a^4 \\ & \qquad\qquad +\ a^4-2a^5+a^6 \\ \hline & =1-2a+3a^2-4a^3+3a^4-2a^5+a^6, \textit{ Ans.}\end{array}$$

7. $(3ax+2a^2-4x^2-5)^2$

$$=9a^2x^2+12a^3x-24ax^3-30ax+4a^4-16a^2x^2-20a^2+16x^4+40x^2+25$$

$$=12a^3x-24ax^3-30ax+4a^4-7a^2x^2-20a^2+16x^4+40x^2+25, \textit{ Ans.}$$

8. $(1-2x-y^2+xy-x^2)^2$

$$\begin{array}{l}=1-4x-2y^2+2xy-2x^2 \\ \qquad\qquad +4x^2+4xy^2-4x^2y+4x^3+y^4-2xy^3+2x^2y^2 \\ \qquad\qquad\qquad\qquad +x^2y^2-2x^3y+x^4 \\ \hline =1-4x-2y^2+2xy+2x^2+4xy^2-4x^2y+4x^3+y^4-2xy^3+3x^2y^2 \\ \qquad\qquad\qquad\qquad -2x^3y+x^4, \textit{ Ans.}\end{array}$$

EVOLUTION.

(**230**, page 166.)

1.
$$\begin{array}{ll} & a^2+2ab+2ac+b^2+2bc+c^2(a+b+c, \textit{ Ans.} \\ & a^2 \\ \hline 2a+b & 2ab+2ac+b^2 \\ & 2ab \qquad +b^2 \\ \hline 2a+2b+c & \qquad 2ac \qquad +2bc+c^2 \\ & \qquad 2ac \qquad +2bc+c^2 \end{array}$$

2.
$$\begin{array}{ll} & a^4-6a^2b+4a^2+9b^2-12b+4(a^2-3b+2, \textit{ Ans.} \\ & a^4 \\ \hline 2a^2-3b & -6a^2b+ \qquad 9b^2 \\ & -6a^2b+ \qquad 9b^2 \\ \hline 2a^2-6b+2 & \qquad 4a^2 \qquad -12b+4 \\ & \qquad 4a^2 \qquad -12b+4. \end{array}$$

3.

$$\begin{array}{ll}
 & x^6+4x^5+2x^4-2x^3+5x^2-2x+1(x^3+2x^2-x+1, \textit{ Ans.} \\
 & x^6 \\ \hline
2x^3+2x^2 & \quad 4x^5+2x^4 \\
 & \quad 4x^5+4x^4 \\ \hline
2x^3+4x^2-x & \qquad -2x^4-2x^3+5x^2 \\
 & \qquad -2x^4-4x^3+\ \ x^2 \\ \hline
2x^3+4x^2-2x+1 & \qquad\qquad 2x^3+4x^2-2x+1 \\
 & \qquad\qquad 2x^3+4x^2-2x+1.
\end{array}$$

4. The two first terms of the root will be $1-a$, by inspection; hence,

$$\begin{array}{ll}
 & 1-2a+3a^2-4a^3+3a^4-2a^5+a^6(1-a+a^2-a^3, \textit{ Ans.} \\
 & 1-2a+\ \ a^2 \\ \hline
2-2a+a^2 & \qquad 2a^2-4a^3+3a^4 \\
 & \qquad 2a^2-2a^3+\ \ a^4 \\ \hline
2-2a+2a^2-a^3 & \qquad\quad -2a^3+2a^4-2a^5+a^6 \\
 & \qquad\quad -2a^3+2a^4-2a^5+a^6
\end{array}$$

5. The square root of the first term is $2a^2b$, and $-12a^3b^2\div 4a^2b=-3ab$; hence we proceed as follows:

$$\begin{array}{ll}
 & \qquad\qquad\qquad\qquad \underline{|2a^2b-3ab+2ab^2, \textit{ Ans.}} \\
 & 4a^4b^2-12a^3b^2+8a^3b^3+9a^2b^2-12a^2b^3+4a^2b^4 \\
 & 4a^4b^2-12a^3b^2+\qquad\quad 9a^2b^2 \\ \hline
4a^2b-6ab+2ab^2 & \qquad\qquad 8a^3b^3 \qquad -12a^2b^3+4a^2b^4 \\
 & \qquad\qquad 8a^3b^3 \qquad -12a^2b^3+4a^2b^4
\end{array}$$

6.

$$\begin{array}{ll}
 & \qquad\qquad\qquad\qquad \underline{|3x^3-5x^2y-4xy^2+6y^3, \textit{ Ans.}} \\
 & 9x^6-30x^5y+\quad x^4y^2+76x^3y^3-44x^2y^4-48xy^5+36y^6 \\
 & 9x^6-30x^5y+25x^4y^2 \\ \hline
6x^3-10x^2y-4xy^2 & \qquad -24x^4y^2+76x^3y^3-44x^2y^4 \\
 & \qquad -24x^4y^2+40x^3y^3+16x^2y^4 \\ \hline
6x^3-10x^2y-8xy^2+6y^3 & \qquad\qquad 36x^3y^3-60x^2y^4-48xy^5+36y^6 \\
 & \qquad\qquad 36x^3y^3-60x^2y^4-48xy^5+36y^6
\end{array}$$

7.
$$\begin{array}{l|l}
 & a^4-6a^2bc+4a^2cd-2a^2d^2+9b^2c^2-12bc^2d+6bcd^2+4c^2d^2-4cd^3+d^4\ \big(a^2-3bc+2cd-d^2,\ \textit{Ans.} \\
 & a^4-6a^2bc \qquad\qquad\qquad +9b^2c^2 \\ \hline
2a^2-6bc+2cd & 4a^2cd-2a^2d^2 \qquad -12bc^2d \\
 & 4a^2cd \qquad\qquad\quad -12bc^2d \qquad +4c^2d^2 \\ \hline
2a^2-6bc+4cd-d^2 & -2a^2d^2 \qquad +6bcd^2 \qquad -4cd^3+d^4 \\
 & -2a^2d^2 \qquad +6bcd^2 \qquad -4cd^3+d^4
\end{array}$$

8.
$$\begin{array}{l|l}
 & a^4-a^3b+\frac{3a^2b^2}{4}-\frac{ab^3}{4}+\frac{b^4}{16}\ \left(a^2-\frac{ab}{2}+\frac{b^2}{4},\ \textit{Ans.}\right. \\
 & a^4-a^3b+\frac{a^2b^2}{4} \\ \hline
2a^2-ab+\frac{b^2}{4} & \frac{a^2b^2}{2}-\frac{ab^3}{4}+\frac{b^4}{16} \\
 & \frac{a^2b^2}{2}-\frac{ab^3}{4}+\frac{b^4}{16}
\end{array}$$

9.
$$\begin{array}{l|l}
 & x^8-6x^5+11x^2-6x^{-1}+x^{-4}\ \big(x^4-3x+x^{-2},\ \textit{Ans.} \\
 & x^8-6x^5+\ 9x^2 \\ \hline
2x^4-6x+x^{-2} & 2x^2-6x^{-1}+x^{-4} \\
 & 2x^2-6x^{-1}+x^{-4}
\end{array}$$

10.
$$\begin{array}{l|l}
 & a^2b^{-2}-10ab^{-1}+27-10a^{-1}b+a^{-2}b^2\ \big(ab^{-1}-5+a^{-1}b,\ \textit{Ans.} \\
 & a^2b^{-2}-10ab^{-1}+25 \\ \hline
2ab^{-1}-10+a^{-1}b & +\ 2-10a^{-1}b+a^{-2}b^2 \\
 & \quad\ 2-10a^{-1}b+a^{-2}b^2
\end{array}$$

In this example we have the last multiplication,

$$2ab^{-1}\times a^{-1}b=2a^0b^0=2.$$

11.
$$\begin{array}{l|l}
 & a^{4m}+6a^{3m}c^n+11a^{2m}c^{2n}+6a^mc^{3n}+c^{4n}\ \big(a^{2m}+3a^mc^n+c^{2n},\ \textit{Ans.} \\
 & a^{4m} \\ \hline
2a^{2m}+3a^mc^n & 6a^{3m}c^n+11a^{2m}c^{2n} \\
 & 6a^{3m}c^n+\ 9a^{2m}c^{2n} \\ \hline
2a^{2m}+6a^mc^n+c^{2n} & 2a^{2m}c^{2n}+6a^mc^{3n}+c^{4n} \\
 & 2a^{2m}c^{2n}+6a^mc^{3n}+c^{4n}
\end{array}$$

SQUARE ROOT OF NUMBERS.

(**234**, page 169.)

1. 72,25(85, *Ans.* (a b over 85)

$a^2=$	64
$2a+b=165$	825
$2ab+b^2=$	825

2. 10,82,41(329, *Ans.* (a b c over 329)

	9
$2a+b=62$	182
	124
$2a+2b+c=649$	5841
	5841

3. 65,12,49(807, *Ans.*

	64
1607	11249
	11249

4. 97,41,69(987, *Ans.*

	81
188	1641
	1504
1967	13769
	13769

5. 5,09,85,64(2258, *Ans.*

	4
42	109
	84
445	2585
	2225
4508	36064
	36064

6. 66,34.10,25(81.45, *Ans.*

	64
161	234
	161
1624	7310
	6496
16285	81425
	81425

7. 18,12,88,60,84(42578, *Ans.*

	16
82	212
	164
845	4888
	4225
8507	66360
	59549
85148	681184
	681184

8. .33,98,89(.583, *Ans.*

	25
108	898
	864
1163	3489
	3489

9. .00,52,41,76(.0724, *Ans.*

	49
142	341
	284
1444	5776
	5776

10. 4,77(21.8403+, *Ans.*

	4
41	77
	41
428	3600
	3424
4364	17600
	17464
436803	1360000
	1310409

11. 11.09(3.33016+, *Ans.*

	9
63	209
	189
663	2000
	1989
66601	110000
	66601
666026	4339900
	3996156

12. The square root of a fraction is the square root of the numerator divided by the square root of the denominator. Hence,

13,69(37

	9
67	469
	469

1,18,81(109

	1
209	1881
	1881

$\frac{37}{109}$, *Ans.*

13. 1,02,03,02,01(10101

	1
201	0203
	201
20201	20201
	20201

$\frac{1}{10101}$, *Ans.*

14. $\frac{245}{720} = \frac{49}{144}$. Hence, $\frac{7}{12}$, *Ans.*

15. $5\frac{4}{7} = 5.57,14,28,57+(2.3604+$, *Ans.*

	4
43	157
	129
466	2814
	2796
27204	182857
	168816

CONTRACTED METHOD.

(**235**, page 171.)

1. 56.00,00,00,0(7.4833147+

	49
144	700
	576
1488	12400
	11904
14963	49600
	44889
14966	4711
	4490
1497	221
	150
150	71
	60
15	11
	11

2. 14.00,00,00(3.7416574

	9
67	500
	469
744	3100
	2976
7481	12400
	7481
7482	4919
	4489
748	430
	374
75	56
	53
7	3
	3

3\. 18.00,00(4.2426+

	16
82	200
	164
844	3600
	3376
848	224
	169
85	55
	51

4\. 19.00,00,00(4.358898+

	16
83	300
	249
865	5100
	4325
8708	77500
	69664
8716	7836
	6973
872	863
	785
87	78
	72

5\. 52.46,30,00(7.2431346+

	49
142	346
	284
1444	6230
	5776
14483	45400
	43449
14486	1951
	1449
1449	502
	435
145	67
	58
14	9
	8

6\. 7.00,00,00,00(2.64575131+

	4
46	300
	276
524	2400
	2096
5285	30400
	26425
52907	397500
	370349
52914	27151
	26457
5291	694
	529
529	165
	159
53	6
	5

7. $5^{\frac{3}{2}}=(5^3)^{\frac{1}{2}}=(125)^{\frac{1}{2}}$

	1,25.00,00,0(11.18034+
	1
21	25
	21
221	400
	211
2228	18900
	18824
2236	76
	67
22	9
	9

(**239**, page 175.)

			$27a^3+108a^2+144a+64(3a+4$, *Ans.*
			$27a^3$
		$27a^2$	$108a^2+144a+64$
$9a+4$	$36a+16$	$27a^2+36a+16$	$108a^2+144a+64$

In this example we have,

Trial divisor,	$27a^2$
First factor of correction,	$9a+4$
Correction of trial divisor,	$36a+16$
Complete divisor,	$27a^2+36a+16$

according to formula (a), (**237.**)

2. $x^6+6x^5-40x^3+96x-64(x^2+2x-4$, *Ans.*

x^6

		$3x^4$	$6x^5-40x^3+96x-64$
$3x^2+2x$	$6x^3+\ 4x^2$	$3x^4+\ 6x^3+4x^2$	$6x^5+12x^4+8x^3$
		$3x^4+12x^3+12x^2$	$-12x^4-48x^3+96x-64$
$3x^2+6x-4$	$-12x^2-24x+16$	$3x^4+12x^3-24x+16$	$-12x^4+48x^3+96x-64$

To form the new trial divisor we have, according to formula (*b*),

$$(3x^4+6x^3+4x^2)+(6x^3+4x^2)+4x^2=3x^4+12x^3+12x^4.$$

3. $8x^6-36x^5+66x^4-63x^3+33x^2-9x+1(2x^2-3x+1$, *Ans.*

$8x^6$

		$12x^4$	$-36x^5+66x^4-63x^3$
$6x^2-3x$	$-18x^3+9x^2$	$12x^4-18x^3+9x^2$	$-36x^5+54x^4-27x^3$
		$12x^4-36x^3+27x^2$	$12x^4-36x^3+33x^2-9x+1$
$6x^2-9x+1$	$6x^2-9x+1$	$12x^4-36x^3+33x^2-9x+1$	$12x^4-36x^3+33x^2-9x+1$

4.

$$a^6+9a^5b+24a^4b^2+9a^3b^3-24a^2b^4+9ab^5-b^6(a^2+3ab-b^2, \text{ Ans.}$$

			a^6
		$3a^4$	$9a^5b+24a^4b^2+9a^3b^3$
$3a^2+3ab$	$9a^3b+9a^2b^2$	$3a^4+9a^3b+9a^2b^2$	$9a^5b+27a^4b^2+27a^3b^3$
		$3a^4+13a^3b+27a^2b^2$	$-3a^4b^2-18a^3b^3-24a^2b^4+9ab^5-b^6$
$3a^2+9ab-b^2,$	$-3a^2b^2-9ab^3+b^4$	$3a^4+18a^3b+24a^2b^2-9ab^3+b^4$	$-3a^4b^2-18a^3b^3-24a^2b^4+9ab^5-b^6$

5.

$$a^9-6a^8+27a^7-74a^6+159a^5-234a^4+257a^3-174a^2+60a-8\ \big|\ a^3-2a^2+5a-2, \text{ Ans.}$$

			a^9
		$3a^6$	$-6a^8+27a^7-74a^6$
$3a^3-2a^2$	$-6a^5+4a^4$	$3a^6-6a^5+4a^4$	$-6a^8+12a^7-8a^6$
		$3a^6-12a^5+12a^4$	$15a^7-66a^6+159a^5-234a^4+257a^3$
$3a^3-6a^2+5a$	$15a^4-30a^3+25a^2$	$3a^6-12a^5+27a^4-30a^3+25a^2$	$15a^7-60a^6+135a^5-150a^4+125a^3$
		$3a^6-12a^5+42a^4-60a^3+75a^2$	$-6a^6+24a^5-84a^4+132a^3-174a^2+60a-8$
$3a^3-6a^2+15a-2,$	$-6a^3+12a^2-30a+4$	$3a^6-12a^5+42a^4-66a^3+87a^2-30a+4$	$-6a^6+24a^5-84a^4+132a^3-174a^2+60a-8$

6.

			$x^9-3x^8+6x^7-10x^6+12x^5-12x^4+10x^3-6x^2+3x-1$ $(x^3-x^2+x-1$, *Ans.*
			x^9
		$3x^6$	$-3x^8+6x^7-10x^6$
$3x^3-x^2$	$-3x^5+x^4$	$3x^6-3x^5+x^4$	$-3x^8+3x^7-\ \ x^6$
		$3x^6-6x^5+3x^4$	$3x^7-\ 9x^6+12x^5-12x^4+10x^3$
$3x^3-3x^2+x$	$3x^4-3x^3+x^2$	$3x^6-6x^5+6x^4-3x^3+x^2$	$3x^7-\ 6x^6+\ 6x^5-\ 3x^4+\ \ x^3$
		$3x^6-6x^5+9x^4-6x^3+3x^2$	$-\ 3x^6+\ 6x^5-\ 9x^4+\ 9x^3-6x^2+3x-1$
$3x^3-3x^2+3x-1$	$-3x^3+3x^2-3x+1$	$3x^6-6x^5+9x^4-9x^3+6x^2-3x+1$	$-\ 3x^6+\ 6x^5-\ 9x^4+\ 9x^3-6x^2-3x-1$

7.

			$\underline{\lvert 2a-ab+3bc}$, *Ans.*
			$8a^3-12a^2b+36a^2bc+6a^2b^2-36a^2b^2c-a^3b^3+54ab^2c^2+9a^2b^3c-27ab^3c^2+27b^3c^3$
			$8a^3$
		$12a^2$	$-12a^2b+36a^2bc+6a^2b^2-36a^2b^2c-a^3b^3$
$6a-ab$	$-6a^2b+a^2b^2$	$12a^2-\ 6a^2b+\ a^2b^2$	$-12a^2b\qquad +6a^2b^2\qquad -a^3b^3$
		$12a^2-12a^2b+3a^2b^2$	$36a^2bc\quad -36a^2b^2c\quad +54ab^2c^2+9a^2b^3c-27ab^3c^2+27b^3c^3$
$6a-3ab+3bc$	$18abc-9ab^2c+9b^2c^2$	$12a^2-12a^2b+3a^2b^2+18abc$ $[-9ab^2c+9b^2c^2$	$36a^2bc\quad -36a^2b^2c\quad +54ab^2c^2+9a^2b^3c-27ab^3c^2+27b^3c^3$

8.

$$x^6-12x^5+\tfrac{195}{4}x^4-70x^3+\tfrac{195}{16}x^2-\tfrac{3}{4}x+\tfrac{1}{16}\ \left(x^2-4x+\tfrac{1}{16},\ \mathit{Ans.}\right.$$

$$\begin{array}{lll|ll}
 & & & x^6 & \\
\hline
 & & 3x^4 & -12x^5+\tfrac{195}{4}x^4-70x^3 & \\
3x^2-4x & -12x^3+16x^2 & 3x^4-12x^3+16x^2 & -12x^5+\tfrac{192}{4}x^4-64x^3 & \\
\hline
 & & 3x^4-24x^3+48x^2 & & \tfrac{3}{4}x^4-6x^3+\tfrac{195}{16}x^2-\tfrac{3}{4}x+\tfrac{1}{16} \\
3x^2-12x+\tfrac{1}{4} & \tfrac{3}{4}x^2-3x+\tfrac{1}{16} & 3x^4-24x^3+\tfrac{195}{4}x^2-3x+\tfrac{1}{16} & & \tfrac{3}{4}x^4-6x^3+\tfrac{195}{16}x^2-\tfrac{3}{4}x+\tfrac{1}{16}
\end{array}$$

9.

$$x^9+6x^8-64x^6-96x^5+192x^4+512x^3-768x-512\ \left|\underline{x^3+2x^2-4x-8,\ \mathit{Ans.}}\right.$$

$$\begin{array}{lll|ll}
 & & & x^9 & \\
\hline
 & & 3x^6 & +6x^8-64x^6-96x^5 & \\
3x^3+2x^2 & 6x^5+4x^4 & 3x^6+6x^5+4x^4 & +6x^8+12x^7+8x^6 & \\
\hline
 & & 3x^6+12x^5+12x^4 & -12x^7-72x^6-96x^5+192x^4+512x^3 & \\
3x^3+6x^2-4x & -12x^4-24x^3+16x^2 & 3x^6+12x^5-24x^3+16x^2 & -12x^7-48x^6+96x^4-64x^3 & \\
\hline
 & & 3x^6+12x^5-12x^4-48x^3+48x^2 & & -24x^6-96x^5+96x^4+576x^3-768x-512 \\
3x^3+6x^2-12x-8 & -24x^3-48x^2+96x+64 & 3x^6+12x^5-12x^4-72x^3+96x+64 & & -24x^6-96x^5+96x^4+576x^3-768x-512
\end{array}$$

In the preceding solutions, we have made a formal application of the rule in each case. This will enable us to extract the cube root of any algebraic expression by direct process; but when the root is a binomial, we can generally find it by simple inspection. Thus, in the first example the cube root of the first term is $3a$, and that of the last term is 4; from this we infer, since the expression has but four terms, that $3a+4$ is the cube root. This is easily verified.

Again, in example 4, the cube root of the first term is a^2; and dividing the second term by $3a^4$ we have $+3ab$ for the second term of the root. We have also for the cube root of the last term, $-b^2$; and dividing the preceding term by $3b^4$, we have $+3ab$ for the term of the root preceding $-b^2$. Hence we should infer that the cube root is $a^2+3ab-b^2$, which might be verified by involution.

CUBE ROOT OF NUMBERS.

(**242**, page 179.)

1. $a^3+3a^2b+3ab^2+b^3=148{,}877$ ($\overset{a\,b}{53}$, *Ans.*

		$a^3=$	125
$3a+b$	$3ab+b^2$	$3a^2$	
153	459	7500	23877
	$3a^2+3ab+b^2=$	7959	23877

2. 571,787 (83, *Ans.*

			512
		19200	59787
243	729	19929	59787

NOTE.—This example may be solved by inspection, and all others in which the root has but two places of figures.

3. 256,047,875 (635, *Ans.*

			216
		10800	40047
183	549	11349	34047
		1190700	6000875
1895	9475	1200175	6000875

4. 354,894,912 (708, *Ans.*
343

		1470000	11894912
2108	16864	1486864	11894912

5. 11,852.352 (22.8, *Ans.*
8

		1200	3845
62	124	1324	2648
		145200	1204352
668	5344	150544	1204352

6. 144,125,083,907 (5243, *Ans.*
125

		7500	19125
152	304	7804	15608
		811200	3517083
1564	6256	817456	3269824
		82372800	247259907
15723	47169	82419969	247259907

7. 128,100,283,921 (5041, *Ans.*
125

		750000	3100283
1504	6016	756016	3024064
		76204800	76219921
15121	15121	76219921	76219921

8. 105,555,569,176 (4726, *Ans,*
64

		4800	41555
127	889	5689	39823
		662700	1732569
1412	2824	665524	1331048
		66835200	401521176
14166	84996	66920196	401521176

9. 731,189,187,729 (9009, *Ans.*

729

		243000000	2189187729
27009	243081	243243081	2189187729

10. 1,762.790,912 (12.08, *Ans.*

1

		300	762
32	64	364	728
		4320000	34790912
3608	28864	4348864	34790912

11. 1,061,520,150,601 (10201, *Ans.*

1

		30000	61520
302	604	30604	61208
		312120000	312150601
30601	30601	312150601	312150601

12. 33,212,361.641,984 (321.44, *Ans.*

27

		2700	6212
92	184	2884	5768
		307200	444361
961	961	308161	308161
		30912300	136200641
9634	38536	30950836	123803344
		3098938800	12397297984
96424	385696	3099324496	12397297984

13. 1,371,737,997,260,631 (111111, *Ans.*

			1
		300	371
31	31	331	331
		36300	40737
331	331	36631	36631
		3696300	4106997
3331	3331	3699631	3699631
		370296300	407366260
33331	33331	370329631	370329631
		37036296300	37036629631
333331	333331	37036629631	37036629631

14. 0.171,467 (0.55555+ *Ans.*

			125
		7500	46467
155	775	8275	41375
		907500	5092000
1655	8275	915775	4578875
		92407500	513125000
16655	83275	92490775	462453875
		9257407500	50671125000
166655	833275	9258240775	46291203875

15. 0.004,235,801,032 (0.1618, *Ans.*

			1
		300	3235
36	216	516	3096
		76800	139801
481	481	77281	77281
		7776300	62520032
4838	38704	7815004	62520032

CONTRACTED METHOD.

(**243**, page 181.)

1.

			1.442249+, *Ans.*
			3.000000
			1
		300	2.000
34	136	436	1744
		58800	256000
424	1696	60496	241984
		62208	14016
432	86	62294	12459
		6238	1557
4	1	6239	1248
		624	309
			250
		62	59
			56

2.

			1.912931+, *Ans.*
			7.000,000(
			1
		300	6000
39	351	651	5859
		108300	141000
571	571	108871	108871
		109443	32129
573	115	109558	21812
		10967	10317
6	5	10972	9875
		1098	442
			329
		110	113
			110

3.

$$\sqrt[3]{156.000,000,000} = 5.38321261+,\ \textit{Ans.}$$

			156.000,000,000
			125
		7500	31000
153	459	7959	23877
		842700	7123000
1598	12784	855484	6843872
		86833200	279128000
16143	48429	86881629	260644887
		86930067	18483113
16149	3230	86933297	17386659
		8693653	1096454
161	16	8693669	869367
		869369	227087
			173874
		86937	53213
			52162
		8694	1051
			869

4.

$$\sqrt[3]{34,786.000} = 32.643859+,\ \textit{Ans.}$$

			34,786.000
			27
		2700	7786
92	148	2884	5768
		307200	2018000
966	5796	312996	1877976
		318828	140024
978	391	319219	127688
		31961	12336
10	3	31964	9589
		3197	2747
			2557
		32	190
			160
		3	30
			28

5.

2.222222 +, *Ans.*
10.973,937
8

		1200	2973
62	124	1324	2648
		145200	325937
662	1324	146524	293048
		147852	33889
666	132	147984	29597
		14812	3292
7	1	14813	2962
		1481	330
			296
		148	34
			30

6.

11.44740066 +, *Ans.*
1,500.101,520
1

		300	500
31	31	331	331
		36300	169101
334	1336	37636	150544
		3898800	18557520
3424	13696	3912496	15649984
		3926208	2907536
3432	2403	3928611	2750028
		393101	157508
34	14	393115	157246
		39313	262
			236
		39	26
			23

7. $1.051963+$, *Ans.*

			1.164,132
			1
		30000	164132
305	1525	31525	157625
		33075	6507
315	32	33107	3311
		3314	3196
			2983
		331	213
			199
		33	13
			10

RADICAL QUANTITIES.

(**247**, page 183.)

1. $\sqrt{75}=\sqrt{25\times 3}=\sqrt{3}$, *Ans.*

2. $\sqrt{98a^2}=\sqrt{49a^2\times 2}=7a\sqrt{2}$, *Ans.*

3. $\sqrt{12x^2y}=\sqrt{4x^2\times 3y}=2x\sqrt{3y}$, *Ans.*

4. $\sqrt[3]{54x^4}=\sqrt[3]{27x^3\times 2x}=3x\sqrt[3]{2x}$, *Ans.*

5. $4\sqrt[3]{108}=4\sqrt[3]{27\times 4}=12\sqrt[3]{4}$, *Ans.*

6. $\sqrt{x^3-a^2x^2}=\sqrt{x^2(x-a^2)}=x\sqrt{x-a^2}$, *Ans.*

7. $6\sqrt[3]{32a^3}=6\sqrt[3]{8a^3\times 4}=12a\sqrt[3]{4}$, *Ans.*

8. $3\sqrt{28a^3x^2}=3\sqrt{4a^2x^2\times 7a}=6ax\sqrt{7a}$, *Ans.*

9. $\sqrt[3]{a^3+a^3b^3}=\sqrt[3]{a^3(1+b^3)}=a\sqrt[3]{1+b^3}$, *Ans.*

10. $(x-y)\sqrt{2x^3-4x^2y+2xy^2}=(x-y)\sqrt{2x(x^2-2xy+y^2)}$
$=(x-y)\sqrt{2x\times(x-y)^2}=(x-y)^2\sqrt{2x}$, *Ans.*

11. $(a-b)\sqrt{2a^2b+4ab^2+2b^3}=(a-b)\sqrt{2b(a^2+2ab+b^2)}$
$=(a-b)\sqrt{2b(a+b)^2}=(a^2-b^2)\sqrt{2b}$, *Ans.*

12. $5b(b^3-b^2)^{\frac{1}{2}}=5b[(b-1)b^2]^{\frac{1}{2}}=5b^2(b-1)^{\frac{1}{2}}$, *Ans.*

13. $(2a^7b^5-3a^5b^7)^{\frac{1}{5}}=[(2a^2-3b^2)a^5b^5]^{\frac{1}{5}}=ab(2a^2-3b^2)^{\frac{1}{5}}$, *Ans.*

14. $\frac{a}{b}(a^4b^5+a^5b^4)^{\frac{1}{3}}=\frac{a}{b}[a^3b^3(ab^2+a^2b)]^{\frac{1}{3}}=a^2(ab^2+a^2b)^{\frac{1}{3}}$, *Ans.*

15. $\sqrt{8a^{2n}x^{4m}}=\sqrt{4a^{2n}x^{4m}\times 2}=2a^nx^{2m}\sqrt{2}$, *Ans.*

16. $\sqrt[3]{a^{4m}c^{5m}}=\sqrt[3]{a^{3m}c^{3m}\times a^mc^{2m}}=a^mc^m\sqrt[3]{a^mc^{2m}}$, *Ans.*

17. $(2x^{2m}y^m-3x^{3m}y^{3m})^{\frac{1}{m}}=[x^{2m}y^m(2-3x^my^{2m})]^{\frac{1}{m}}=x^2y(2-3x^my^{2m})^{\frac{1}{m}}$,
Ans.

18. $a^{-m}c(a^{mn}c^{2n}-a^{2mn}c^n)^{\frac{1}{n}}=a^{-m}c[a^{mn}c^n(c^n-a^{mn})]^{\frac{1}{n}}=c^2(c^n-a^{mn})^{\frac{1}{n}}$,
Ans.

(**248**, page 184.)

3. $\sqrt[3]{\frac{135}{32}}=\sqrt[3]{\frac{270}{64}}=\sqrt[3]{\frac{27}{64}\times 10}=\frac{3}{4}\sqrt[3]{10}$, *Ans.*

4. $\sqrt[3]{\frac{25}{9}}=\sqrt[3]{\frac{75}{27}}=\frac{1}{3}\sqrt[3]{75}$, *Ans.*

5. $\sqrt{\frac{50}{147}}=\sqrt{\frac{150}{441}}=\sqrt{\frac{25}{441}\times 6}=\frac{5}{21}\sqrt{6}$, *Ans.*

6. $2\sqrt{\frac{2a}{3}}=2\sqrt{\frac{6a}{9}}=\frac{2}{3}\sqrt{6a}$, *Ans.*

7. $\frac{5}{6}\sqrt{\frac{72}{245}}=\frac{5}{6}\sqrt{\frac{36}{49}\times\frac{2}{5}}=\frac{5}{6}\sqrt{\frac{36}{49}\times\frac{1}{25}\times 10}=\frac{1}{7}\sqrt{10}$, *Ans.*

8. $\frac{x}{a}\sqrt{\frac{a^3b}{x^2y^2}}=\frac{x}{a}\sqrt{\frac{a^2}{x^2y^2}\times ab}=\frac{1}{y}\sqrt{ab}$, *Ans.*

(**249**, page 185.)

3. $(a-cz)=[(a-cz)^4]^{\frac{1}{4}}=(a^4-4a^3cz+6a^2c^2z^2-4ac^3z^3+c^4z^4)^{\frac{1}{4}}$, *Ans.*

6. $(a-2b)\sqrt{2a}=\sqrt{2a(a-2b)^2}=\sqrt{2a(a^2-4ab+4b^2)}$
$=\sqrt{2a^3-8a^2b+8ab^2}$, *Ans.*

(**250**, page 187.)

4. The least common multiple of 1, 2, 3, and 4 is 12; whence by Rule II,

$\sqrt[12]{a^{12}}$, $\sqrt[12]{a^6c^6}$, $\sqrt[12]{a^8x^4}$, $\sqrt[12]{a^3c^3}$, *Ans.*

8. The least common multiple of 2, m and n is $2mn$; whence,

$\sqrt[2mn]{a^{mn}x^{mn}}$, $\sqrt[2mn]{x^{2n}y^{2n}}$, $\sqrt[2mn]{c^{2m}x^{2m}}$, *Ans.*

ADDITION OF RADICALS.

(**251**, page 188.)

1. $\sqrt{16a^2x}=4a\sqrt{x}$
$\sqrt{4a^2x}=2a\sqrt{x}$
$6a\sqrt{x}$, *Ans.*

2. $\sqrt{32}=4\sqrt{2}$
$\sqrt{72}=6\sqrt{2}$
$\sqrt{128}=8\sqrt{2}$
$18\sqrt{2}$, *Ans.*

3. $\sqrt[3]{40}=2\sqrt[3]{5}$
$\sqrt[3]{135}=3\sqrt[3]{5}$
$\sqrt[3]{625}=5\sqrt[3]{5}$
$10\sqrt[3]{5}$, *Ans.*

4. $\sqrt[3]{108}=3\sqrt[3]{4}$
$9\sqrt[3]{4}$
$\sqrt[3]{1372}=7\sqrt[3]{4}$
$19\sqrt[3]{4}$, *Ans.*

5.
$$\sqrt{\tfrac{1}{2}}=\tfrac{1}{2}\sqrt{2}$$
$$\sqrt{\tfrac{2}{9}}=\tfrac{1}{3}\sqrt{2}$$
$$\sqrt{\tfrac{1}{18}}=\tfrac{1}{6}\sqrt{2}$$
$$\sqrt{2},\ \textit{Ans.}$$

6.
$$\sqrt[3]{\tfrac{81}{8}}=\tfrac{3}{2}\sqrt[3]{3}$$
$$\sqrt[3]{\tfrac{81}{64}}=\tfrac{3}{4}\sqrt[3]{3}$$
$$\sqrt[3]{\tfrac{375}{216}}=\tfrac{5}{6}\sqrt[3]{3}$$
$$\tfrac{37}{12}\sqrt[3]{3},\ \textit{Ans.}$$

7.
$$\tfrac{3}{5}\sqrt{\tfrac{2}{3}}=\tfrac{1}{5}\sqrt{6}$$
$$\tfrac{3}{4}\sqrt{\tfrac{25}{6}}=\tfrac{5}{8}\sqrt{6}$$
$$\tfrac{7}{8}\sqrt{\tfrac{96}{25}}=\tfrac{7}{10}\sqrt{6}$$
$$\tfrac{61}{40}\sqrt{6},\ \textit{Ans.}$$

8.
$$3\sqrt{abm^2}=3m\sqrt{ab}$$
$$m\sqrt{4ab}=2m\sqrt{ab}$$
$$\sqrt{25abm^2}=5m\sqrt{ab}$$
$$10m\sqrt{ab},\ \textit{Ans.}$$

9.
$$2a\sqrt{c^2x-c^2y}=2ac\sqrt{x-y}$$
$$3c\sqrt{a^2x-a^2y}=3ac\sqrt{x-y}$$
$$5\sqrt{a^2c^2x-a^2c^2y}=5ac\sqrt{x-y}$$
$$10ac\sqrt{x-y},\ \textit{Ans.}$$

10.
$$\sqrt{20a^2m-20acm+5mc^2}=\sqrt{5m(4a^2-4ac+c^2)}=(2a-c)\sqrt{5m}$$
$$\sqrt{20mc^2-60acm+45a^2m}=\sqrt{5m(4c^2-12ac+9a^2)}=(2c-3a)\sqrt{5m}$$
$$(c-a)\sqrt{5m},\ \textit{Ans.}$$

11.
$$3\sqrt[3]{cx^3}=3x\sqrt[3]{c}$$
$$\sqrt[3]{ax^3}=x\sqrt[3]{a}$$
$$2\sqrt[3]{ax^3}=2x\sqrt[3]{a}$$
$$3x\sqrt[3]{c}+3x\sqrt[3]{a}=3x(\sqrt[3]{a}+\sqrt[3]{c}),\ \textit{Ans.}$$

12.
$$5a(cx^3-dx^3)^{\frac{1}{3}}=5ax(c-d)^{\frac{1}{3}}$$
$$2x(a^3d-a^3c)^{\frac{1}{3}}=2ax(d-c)^{\frac{1}{3}}=-2ax(c-d)^{\frac{1}{3}}$$
$$3ax(c-d)^{\frac{1}{3}},\ \textit{Ans.}$$

NOTE.—In transforming the second quantity, observe that the cube root of a *negative* quantity is *negative*.

13. $$\sqrt{\frac{a^2(a-b)}{a+b}}=\sqrt{\frac{a^2}{(a+b)^2}(a^2-b^2)}=\frac{a}{a+b}\sqrt{a^2-b^2}$$

$$\sqrt{\frac{b^2(a+b)}{a-b}}=\sqrt{\frac{b^2}{(a-b)^2}(a^2-b^2)}=\frac{b}{a-b}\sqrt{a^2-b^2}$$

$$(a^2-3b^2)\sqrt{\frac{1}{a^2-b^2}}=(a^2-3b^2)\sqrt{\frac{a^2-b^2}{(a^2-b^2)^2}}=\frac{a^2-3b^2}{a^2-b^2}\sqrt{a^2-b^2}$$

Hence the sum is,

$$\left(\frac{a}{a+b}+\frac{b}{a-b}+\frac{a^2-3b^2}{a^2-b^2}\right)\sqrt{a^2-b^2}=\left(\frac{a^2-ab+ab+b^2+a^2-3b^2}{a^2-b^2}\right)\sqrt{a^2-b^2}$$

$$=2\sqrt{a^2-b^2}, \textit{ Ans.}$$

14. $$\sqrt{(1+a)^{-1}}=\sqrt{\frac{1}{(1+a)^2}(1+a)}=\frac{1}{1+a}\sqrt{1+a}$$

$$\sqrt{a^2(1+a)^{-1}}=\sqrt{\frac{a^2}{(1+a)^2}(1+a)}=\frac{a}{1+a}\sqrt{1+a}$$

$$a\sqrt{(1+a)(1-a)^{-2}}=a\sqrt{\frac{1}{(1-a)^2}(1+a)}=\frac{a}{1-a}\sqrt{1+a}$$

Hence the sum is

$$\left(\frac{1}{1+a}+\frac{a}{1+a}+\frac{a}{1-a}\right)\sqrt{1+a}=\frac{1+a}{1-a^2}\sqrt{1+a}=\frac{1}{1-a}\sqrt{1+a}$$

$$=\frac{\sqrt{1+a}}{1-a}, \textit{ Ans.}$$

SUBTRACTION OF RADICALS.

(**252**, page 189.)

1. $4\sqrt{135}=12\sqrt{15}$
$2\sqrt{60}\ =\ 4\sqrt{15}$
$8\sqrt{15}$, *Ans.*

2. $\sqrt{75}=5\sqrt{3}$
$\sqrt{50}=5\sqrt{2}$
$5(\sqrt{3}-\sqrt{2})$, *Ans.*

3. $3\sqrt{16a^4b}=12a^2\sqrt{b}$

$3\sqrt{a^2b}=3a\sqrt{b}$

$(12a^2-3a)\sqrt{b}$, *Ans.*

4. $\frac{1}{2}\sqrt[3]{\frac{297}{8}}=\frac{1}{2}\sqrt[3]{\frac{27}{8}\cdot 11}=\frac{3}{4}\sqrt[3]{11}$

$\frac{1}{4}\sqrt[3]{\frac{3773}{125}}=\frac{1}{4}\sqrt[3]{\frac{343}{125}\cdot 11}=\frac{7}{20}\sqrt[3]{11}$

$\frac{2}{5}\sqrt[3]{11}$, *Ans.*

5. $$\frac{2}{3}\sqrt{\frac{490a^2}{338}}=\frac{2}{3}\sqrt{\frac{490a^2}{169}\cdot 5}=\frac{14a}{39}\sqrt{5}$$

$$\frac{a}{13}\sqrt{\frac{361}{5}}=\frac{a}{13}\sqrt{\frac{361}{25}\cdot 5}=\frac{19a}{65}\sqrt{5}$$

Hence the difference is

$$\left(\frac{14a}{39}-\frac{19a}{65}\right)\sqrt{5}=\frac{a}{15}\sqrt{5}, \textit{ Ans.}$$

6. $$(a^2c^3-3c^3x)^{\frac{1}{3}}=c(a^2-3x)^{\frac{1}{3}}$$

$$2(a^2d^3-3d^3x)^{\frac{1}{3}}=2d(a^2-3x)^{\frac{1}{3}}$$

$$(c-2d)(a^2-3x)^{\frac{1}{3}}, \textit{ Ans.}$$

7. $(a^3-ab^2+a^2b-b^3)^{\frac{1}{2}}=\{a^3-b^3+ab(a-b)\}^{\frac{1}{2}}$

$=\{(a^2+ab+b^2+ab)(a-b)\}^{\frac{1}{2}}=(a+b)(a-b)^{\frac{1}{2}}$

$(a^3-3a^2b+3ab^2-b^3)^{\frac{1}{2}}=\ (a-b)^{\frac{3}{2}}\qquad =(a-b)(a-b)^{\frac{1}{2}}$

$2b(a-b)^{\frac{1}{2}}$, *Ans.*

8. $$a\sqrt{\frac{b^2x+b^2}{x-1}}=ab\sqrt{\frac{x+1}{x-1}}=ab\sqrt{\frac{x^2-1}{(x-1)^2}}=\frac{ab}{x-1}\sqrt{x^2-1}$$

$$b\sqrt{\frac{a^2x-a^2}{x+1}}=ab\sqrt{\frac{x-1}{x+1}}=ab\sqrt{\frac{x^2-1}{(x+1)^2}}=\frac{ab}{x+1}\sqrt{x^2-1}$$

$$\frac{2ab}{x^2-1}\sqrt{x^2-1}, \textit{ Ans.}$$

MULTIPLICATION OF RADICALS.

(**253**, page 191.)

1. $5\sqrt{5} \times 3\sqrt{8} = 15\sqrt{40} = 15\sqrt{4 \times 10} = 30\sqrt{10}$, *Ans.*

2. $4\sqrt{12} \times 3\sqrt{2} = 12\sqrt{24} = 12\sqrt{4 \times 6} = 24\sqrt{6}$, *Ans.*

3. $3\sqrt{2} \times 2\sqrt{8} = 6\sqrt{16} = 24$, *Ans.*

4. $2\sqrt{5} \times 2\sqrt{10} \times 3\sqrt{6} = 12\sqrt{300} = 12\sqrt{100 \times 3} = 120\sqrt{3}$, *Ans.*

5. $2\sqrt[3]{14} \times 3\sqrt[3]{4} = 6\sqrt[3]{56} = 6\sqrt[3]{8 \times 7} = 12\sqrt[3]{7}$, *Ans.*

6. $5c\sqrt{ax} \times c\sqrt[3]{a^2} \times \sqrt[3]{ax^2} = 5c\sqrt[6]{a^3x^3} \times c\sqrt[6]{a^4} \times \sqrt[6]{a^2x^4} = 5c^2\sqrt[6]{a^9x^7}$

$= 5ac^2x\sqrt[6]{a^3x}$, *Ans.*

7. $(xy)^{\frac{1}{2}} \times (xz)^{\frac{2}{3}} + (yz)^{\frac{3}{4}} = (x^6y^6)^{\frac{1}{12}} \times (x^8z^8)^{\frac{1}{12}} \times (y^9z^9)^{\frac{1}{12}}$

$= (x^{14}y^{15}z^{17})^{\frac{1}{12}} = xyz(x^2y^3z^5)^{\frac{1}{12}}$, *Ans.*

8. $(x-y)^{\frac{2}{3}} \times (x+y)^{\frac{3}{4}} = [(x-y)^8(x+y)^9]^{\frac{1}{12}} = \sqrt[12]{(x-y)^8(x+y)^9}$

$= \sqrt[12]{\{(x-y)(x+y)\}^8(x+y)}$

$= \sqrt[12]{(x^2-y^2)^8(x+y)}$, *Ans.*

9. $\sqrt[3]{15} \times \sqrt{10} = \sqrt[6]{225} \times \sqrt[6]{1000} = \sqrt[6]{225000}$, *Ans.*

10. $\frac{a}{b}\sqrt{\frac{x}{y}} \times \frac{y}{x}\sqrt[3]{\frac{b^2}{a^2}} \times \sqrt[3]{\frac{bx^2}{ay^2}} = \frac{ay}{bx}\sqrt[6]{\frac{x^3}{y^3}} \times \sqrt[6]{\frac{b^4}{a^4}} \times \sqrt[6]{\frac{b^2x^4}{a^2y^4}}$

$= \frac{ay}{bx}\sqrt[6]{\frac{b^6x^7}{a^6y^7}} = \sqrt[6]{\frac{x}{y}}$, *Ans.*

11. If we square the two first factors under the radical sign, $\sqrt{}$, we may extend the sign, $\sqrt[4]{}$, over the whole product; thus,

$$\sqrt[4]{\frac{a^2x^4}{(a+x)^4}\times\frac{b^2(a^2-x^2)^4}{x^2}\times\frac{a^4c}{(a-x)^4}}=\sqrt[4]{\frac{(a^2-x^2)^4a^4}{(a+x)^4(a-x)^4x^4}a^2b^2c}$$

$$=\frac{a}{x}\sqrt[4]{\frac{(a^2-x^2)^4}{(a^2-x^2)^4}a^2b^2c}=\frac{a}{x}\sqrt[4]{a^2b^2c},\ \textit{Ans.}$$

DIVISION OF RADICALS.

(**254**, page 192.)

3. $\dfrac{\sqrt[3]{20a^2d}}{\sqrt{15ad}}=\dfrac{\sqrt[6]{400a^4d^2}}{\sqrt[6]{3375a^3d^3}}=\sqrt[6]{\dfrac{400a^4d^2}{3375a^3d^3}}=\sqrt[6]{\dfrac{16a}{135d}}$, *Ans.*

4. $\dfrac{(a^2b^2d^2)^{\frac{1}{6}}}{(d^2)^{\frac{1}{6}}}=(a^2b^2)^{\frac{1}{6}}=(ab)^{\frac{1}{3}}$, *Ans.*

5. $\dfrac{(16a^3-12a^2x)^{\frac{1}{2}}}{(4a^2)^{\frac{1}{2}}}=(4a-3x)^{\frac{1}{2}}$, *Ans.*

6. $\dfrac{45}{3\sqrt{5}}=\dfrac{3\cdot5}{\sqrt{5}}=\dfrac{3(5^2)^{\frac{1}{2}}}{(5)^{\frac{1}{2}}}=3\sqrt{5}$, *Ans.*

7. $\dfrac{(ab^2c^2)^{\frac{1}{3}}}{(a^2b^3c^4)^{\frac{1}{5}}}=\dfrac{(a^5b^{10}c^{10})^{\frac{1}{15}}}{(a^6b^9c^{12})^{\frac{1}{15}}}=\sqrt[15]{\dfrac{b}{ac^2}}$, *Ans.*

8. $\dfrac{12c^2(a-x)^{\frac{3}{4}}}{4c(a-x)^{\frac{2}{3}}}=\dfrac{3c[(a-x)^9]^{\frac{1}{12}}}{[(a-x)^8]^{\frac{1}{12}}}=3c(a-x)^{\frac{1}{12}}$, *Ans.*

9. $\dfrac{(a^3c)^{\frac{1}{m}}}{(ac^3)^{\frac{1}{n}}}=\dfrac{(a^{3n}c^n)^{\frac{1}{mn}}}{(a^mc^{3m})^{\frac{1}{mn}}}=(a^{3n-m}c^{n-3m})^{\frac{1}{mn}}$, *Ans.*

10. $\dfrac{\sqrt{a}}{\sqrt[3]{x}} \times \dfrac{\sqrt[3]{x^3}}{\sqrt[4]{a^3}} = \dfrac{\sqrt[4]{a^2} \times \sqrt[3]{x}}{\sqrt[4]{a^3}} = \dfrac{\sqrt[12]{a^6x^4}}{\sqrt[12]{a^9}} = \sqrt[12]{\dfrac{x^4}{a^3}}$, *Ans.*

11. $\dfrac{\sqrt[4]{a^2b-ab^2}}{\sqrt{ab}} = \dfrac{\sqrt[4]{a^2b-ab^2}}{\sqrt[4]{a^2b^2}} = \dfrac{\sqrt[4]{a^2b^2(a^2b-ab^2)}}{\sqrt[4]{a^4b^4}} = \dfrac{1}{ab}\sqrt[4]{a^4b^3-a^3b^4}$, *Ans.*

(**258**, page 194.)

1. $(\sqrt[5]{2a^2})^3 = \sqrt[5]{8a^6} = \sqrt[5]{8a \times a^5} = a\sqrt[5]{8a}$, *Ans.*

2. $(\sqrt[3]{x^2y^2})^2 = \sqrt[3]{x^4y^4} = \sqrt[3]{x^3y^3 \times xy} = xy\sqrt[3]{xy}$, *Ans.*

3. $(3\sqrt[6]{4a^5c})^4 = 81\sqrt[6]{256a^{20}c^4} = 81\sqrt[3]{\sqrt{256a^{20}c^4}} = 81\sqrt[3]{16a^{10}c^2}$
$= 81\sqrt[3]{8a^9 \times 2ac^2} = 162a^3\sqrt[3]{2ac^2}$, *Ans.*

4. $[(a-b)^{\frac{3}{4}}]^2 = (a-b)^{\frac{6}{4}} = (a-b)^{\frac{3}{2}}$, *Ans.*

5. $(\sqrt[10]{12ab^2})^5 = \sqrt{12ab^2} = \sqrt{4b^2 \times 3a} = 2b\sqrt{3a}$, *Ans.*

6. $\left(\sqrt[12]{c(a-x)^2}\right)^8 = \left(\sqrt[3]{c(a-x)^2}\right)^2 = \sqrt[3]{c^2(a-x)^4} = (a-x)\sqrt[3]{c^2(a-x)}$, *Ans.*

7. $(ax\sqrt{ax})^4 = a^4x^4 \cdot a^2x^2 = a^6x^6$, *Ans.*

8. $\left(\sqrt[3]{x^2y^2(x-y)}\right)^2 = \sqrt[3]{x^4y^4(x-y)^2} = xy\sqrt[3]{xy(x-y)^2}$, *Ans.*

10. $\left(\dfrac{a}{x}\sqrt[10]{96cx^6}\right)^2 = \dfrac{a^2}{x^2}\sqrt[5]{96cx^6} = \dfrac{a^2}{x^2}\sqrt[5]{32x^5 \times 3cx} = \dfrac{2a^2}{x}\sqrt[5]{3cx}$, *Ans.*

(**259**, page 196.)

2. $\sqrt[3]{a\sqrt[4]{a^2x^3}} = \sqrt[3]{\sqrt[4]{a^6x^3}} = \sqrt[4]{\sqrt[3]{a^6x^3}} = \sqrt[4]{a^2x}$, *Ans.*

3. $\sqrt[4]{2\sqrt[3]{98}} = \sqrt[4]{\sqrt[3]{8 \times 98}} = \sqrt[6]{\sqrt{784}} = \sqrt[6]{28}$, *Ans.*

4. Putting the fifth power of the coefficient under the radical sign, we have

$$\sqrt{\sqrt[5]{\tfrac{32}{243}\cdot 486}}=\sqrt[5]{\sqrt{64}}=\sqrt[5]{8},\ \textit{Ans.}$$

6. $\sqrt[3]{5\sqrt{5}}=\sqrt[3]{\sqrt{125}}=\sqrt{\sqrt[3]{125}}=\sqrt{5}$, *Ans.*

7. $\left[\left(\frac{a^2x^6}{c^6y^2}\right)^{\frac{2}{5}}\right]^{\frac{1}{6}}=\left[\left(\frac{a^2b^6}{c^6y^2}\right)^{\frac{1}{3}}\right]^{\frac{1}{5}}$; dividing out the factor 2 in the exponents. Hence, $\left(\frac{ax^3}{c^3y}\right)^{\frac{1}{5}}$, *Ans.*

8. $\sqrt[4]{\frac{4}{9}\sqrt[3]{\frac{4}{9}}}=\sqrt[4]{\sqrt[3]{\frac{4^4}{9^4}}}=\sqrt[3]{\sqrt[4]{\frac{4^4}{9^4}}}=\sqrt[3]{\frac{4}{9}}=\sqrt[3]{\frac{1}{27}.12}=\frac{1}{3}\sqrt[3]{12}$, *Ans.*

(**260**, page 197.)

3. $\sqrt[4]{1296}=\sqrt{\sqrt{1296}}=\sqrt{36}=6$, *Ans.*

4. $\sqrt[6]{177978515625}=\sqrt[3]{\sqrt{177978515625}}$
$=\sqrt[3]{421875}=75$, *Ans.*

5. $\sqrt[6]{191102976}=\sqrt[3]{\sqrt{191102976}}=\sqrt[3]{13824}=24$, *Ans.*

6. $\sqrt[8]{65536}=\sqrt{\sqrt{(65536)^{\frac{1}{2}}}}=\sqrt{(256)^{\frac{1}{2}}}=(16)^{\frac{1}{2}}=4$, *Ans.*

7. We first take the square root, which is
$$a^2-4ab+4b^2;$$
then the square root of this is, $a-2b$, *Ans.*

8. The square root of the expression is
$$a^6+3a^4b+3a^2b^2+b^3;$$
and the cube root of this is a^2+b, *Ans.*

GENERAL THEORY OF EXPONENTS.

(**261**, page 199.)

6. $a^2b^{\frac{1}{4}} \times a^{\frac{1}{2}}b^{\frac{1}{4}} = a^{\frac{5}{2}}b^{\frac{1}{2}} = a^2\sqrt{ab}$, *Ans.*

7. Adding the exponents we have

$$\tfrac{1}{4}+\tfrac{1}{2}+\tfrac{2}{3}-\tfrac{3}{4}=\tfrac{2}{3}.\quad \text{Hence, } \sqrt[3]{a^2}, \textit{ Ans.}$$

8. $\dfrac{a^{\frac{1}{2}}c^{\frac{2}{3}}}{a^{\frac{2}{3}}c^{\frac{1}{3}}} = a^{\frac{1}{2}-\frac{2}{3}} \times c^{\frac{2}{3}-\frac{1}{3}} = a^{-\frac{1}{6}} \times c^{\frac{1}{3}} = \left(\dfrac{c^2}{a}\right)^{\frac{1}{6}}$, *Ans.*

9. $\dfrac{\left(x^{\frac{3}{4}}\right)^{\frac{2}{3}}}{\left(x^{\frac{1}{2}}\right)^{\frac{4}{3}}} = \dfrac{x^{\frac{1}{2}}}{x^{\frac{2}{3}}} = x^{\frac{1}{2}-\frac{2}{3}} = x^{-\frac{1}{6}} = \left(\dfrac{1}{x}\right)^{\frac{1}{6}}$, *Ans.*

10. $\left(a^{\frac{1}{2}}-a^{\frac{1}{4}}\right)\left(a^{\frac{1}{4}}+1\right) = a^{\frac{3}{4}}+a^{\frac{1}{2}}-a^{\frac{1}{2}}-a^{\frac{1}{4}} = a^{\frac{3}{4}}-a^{\frac{1}{4}}$, *Ans.*

11. $\left(2\sqrt[3]{x^2}+\sqrt{xy}\right)\left(3\sqrt[3]{x}-\sqrt{xy}\right) = 6\sqrt[3]{x^3}-2\sqrt[3]{x^2}\cdot\sqrt{xy}+3\sqrt[3]{x}\cdot\sqrt{xy}-xy$

$$= 6x-2\sqrt[6]{x^4}\cdot\sqrt[6]{x^3}\cdot\sqrt[6]{y^3}+3\sqrt[6]{x^2}\cdot\sqrt[6]{x^3}\cdot\sqrt[6]{y^3}-xy,$$

$$= 6x-2\sqrt[6]{x^7y^3}+3\sqrt[6]{x^5y^3}-xy, \textit{ Ans.}$$

12.
$$\begin{array}{llll}
a^{\frac{1}{2}}-2a^{-\frac{1}{4}} & +\ a^{-1} & & \\
a^{\frac{1}{4}}-\ a^{-\frac{1}{2}} & & & \\
\hline
a^{\frac{3}{4}}-2a^{0} & +\ a^{-\frac{3}{4}} & & \\
\quad -\ a^{0} & +2a^{-\frac{3}{4}} & -a^{-\frac{3}{2}} & \\
\hline
a^{\frac{3}{4}}-3 & +3a^{-\frac{3}{4}} & -a^{-\frac{3}{2}}, & \textit{Ans.}
\end{array}$$

13.
$$\begin{array}{l|l}
a-b & \sqrt{a}+\sqrt{b} \\
\cline{2-2}
a+\sqrt{ab} & \sqrt{a}-\sqrt{b}, \textit{ Ans.} \\
\hline
-\sqrt{ab}-b & \\
-\sqrt{ab}-b &
\end{array}$$

14.
$$\begin{array}{l|l} a^{\frac{5}{6}}-2a^{\frac{1}{2}}+a^{\frac{1}{6}} & a^{\frac{1}{3}}-1 \\ \hline \underline{a^{\frac{5}{6}}-\ a^{\frac{1}{2}}} & a^{\frac{1}{2}}-a^{\frac{1}{6}},\ \textit{Ans.} \\ \quad -\ a^{\frac{1}{2}}+a^{\frac{1}{6}} & \\ \quad -\ a^{\frac{1}{2}}+a^{\frac{1}{6}} & \end{array}$$

15.
$$\begin{array}{l} a^{\frac{5}{2}}+a^{2}b^{\frac{2}{3}}+a^{\frac{3}{2}}b^{\frac{4}{3}}+ab^{2}+a^{\frac{1}{2}}b^{\frac{8}{3}}+b^{\frac{10}{3}} \\ \underline{a^{\frac{1}{2}}-b^{\frac{2}{3}}} \\ a^{3}+a^{\frac{5}{2}}b^{\frac{2}{3}}+a^{2}b^{\frac{4}{3}}+a^{\frac{3}{2}}b^{2}+ab^{\frac{8}{3}}+a^{\frac{1}{2}}b^{\frac{10}{3}} \\ \underline{\qquad -a^{\frac{5}{2}}b^{\frac{2}{3}}-a^{2}b^{\frac{4}{3}}-a^{\frac{3}{2}}b^{2}-ab^{\frac{8}{3}}-a^{\frac{1}{2}}b^{\frac{10}{3}}-b^{4}} \\ a^{3}-b^{4},\ \textit{Ans.} \end{array}$$

16.
$$\begin{array}{l|l} x^{\frac{4}{3}}+\ x^{\frac{2}{3}}a^{\frac{2}{3}}+a^{\frac{4}{3}} & x^{\frac{2}{3}}+x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}} \\ \hline \underline{x^{\frac{4}{3}}+xa^{\frac{1}{3}}+x^{\frac{2}{3}}a^{\frac{2}{3}}} & x^{\frac{2}{3}}-x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}},\ \textit{Ans.} \\ \quad -xa^{\frac{1}{3}}+a^{\frac{4}{3}} & \\ \quad \underline{-xa^{\frac{1}{3}}-x^{\frac{2}{3}}a^{\frac{2}{3}}-x^{\frac{1}{3}}a} & \\ \qquad\qquad x^{\frac{2}{3}}a^{\frac{2}{3}}+x^{\frac{1}{3}}a+a^{\frac{4}{3}} & \\ \qquad\qquad x^{\frac{2}{3}}a^{\frac{2}{3}}+x^{\frac{1}{3}}a+a^{\frac{4}{3}} & \end{array}$$

17. $(a^{\frac{1}{3}}\cdot a^{\frac{5}{7}})^{\frac{7}{11}}=(a^{\frac{22}{21}})^{\frac{7}{11}}=a^{\frac{22}{21}\cdot\frac{7}{11}}=a^{\frac{2}{3}}$, *Ans.*

18. $\left(\frac{1\pm\sqrt{5}}{2}\right)^{3}=\frac{1\pm3\sqrt{5}+15\pm5\sqrt{5}}{8}=2\pm\sqrt{5}$, *Ans.*

19. Raising each factor of the numerator to the power of nr, and each factor of the denominator to the power of ms, the expression becomes

$$\frac{a^{mn}\times a^{rs}\times c^{mr}}{c^{mn}\times c^{rs}\times a^{mr}}=\frac{a^{mn+rs-mr}}{c^{mn+rs-mr}}=\left(\frac{a}{c}\right)^{mn+rs-mr}.\quad \textit{Ans.}$$

20. $$\left\{\frac{2\sqrt{3}\cdot2\sqrt[3]{108}}{3\sqrt[3]{72}\cdot3\sqrt{3}}\right\}^{\frac{1}{2}}=\left\{\frac{2\sqrt{3}\cdot6\sqrt[3]{4}}{6\sqrt[3]{9}\cdot3\sqrt{3}}\right\}^{\frac{1}{2}}=\left(\frac{2}{3}\sqrt[3]{\frac{4}{9}}\right)^{\frac{1}{2}}=\left(\frac{2}{3}\sqrt[3]{\left(\frac{2}{3}\right)^2}\right)^{\frac{1}{2}}$$

$$\left(\frac{2}{3}\sqrt[3]{\left(\frac{2}{3}\right)^3\left(\frac{3}{2}\right)}\right)^{\frac{1}{2}}=\frac{2}{3}\left(\sqrt[3]{\frac{3}{2}}\right)^{\frac{1}{2}}=\frac{2}{3}\sqrt[6]{\frac{3}{2}},\ \textit{Ans.}$$

21. $$\left\{\frac{\frac{4}{5}\sqrt[3]{\frac{1}{5}}-2\sqrt{\frac{1}{2}}}{\frac{1}{2}\sqrt[3]{5}+\frac{5}{4}\sqrt[4]{2}}\right\}^{\frac{1}{2}}=\left\{\frac{\frac{1}{25}\left(4\sqrt[3]{25}-25\sqrt{2}\right)}{\frac{1}{4}\left(2\sqrt[3]{5}+5\sqrt[4]{2}\right)}\right\}^{\frac{1}{2}}$$

$$=\frac{2}{5}\left\{\frac{\left(2\sqrt[3]{5}-5\sqrt[4]{2}\right)\left(2\sqrt[3]{5}+5\sqrt[4]{2}\right)}{2\sqrt[3]{5}+5\sqrt[4]{2}}\right\}^{\frac{1}{2}}$$

$$=\tfrac{2}{5}\left(2\sqrt[3]{5}-5\sqrt[4]{2}\right)^{\frac{1}{2}},\ \textit{Ans.}$$

22. Performing the multiplications indicated in the numerator and denominator, the expression becomes

$$\frac{a-b}{\sqrt{a}-\sqrt{b}}=\sqrt{a}+\sqrt{b},\ \textit{Ans.}$$

23. Omitting the exponent, the numerator is

$$(\sqrt{5}+2)(\sqrt[4]{5}+\sqrt{2})(\sqrt[4]{5}-\sqrt{2})=(\sqrt{5}+2)(\sqrt{5}-2)=5-4=1;$$

and the denominator,

$$\left(\sqrt{13}+3\right)\left(\sqrt[4]{13}+\sqrt{3}\right)\left(\sqrt[4]{13}-\sqrt{3}\right)=\left(\sqrt{13}+3\right)\left(\sqrt{13}-3\right)=13-9=4;$$

and taking the square root, $\frac{1}{2}$, *Ans.*

IMAGINARY QUANTITIES.

(**268**, page 203.)

10.

$$\left(a+\sqrt{-c}\right)^4=a^4+4a^3\sqrt{-c}+6a^2\left(\sqrt{-c}\right)^2+4a\left(\sqrt{-c}\right)^3+\left(\sqrt{-c}\right)^4.$$

$$=a^4+c^2-6a^2c+(4a^3-4ac)\sqrt{-c},\ \textit{Ans.}$$

(**200—203**)

11.
$$
\begin{array}{l}
a-\sqrt{-a}\,)\ a^2+\sqrt{-a}\ \left(a+\sqrt{-a}-1,\ \textit{Ans.}\right. \\
\qquad\qquad\ \underline{a^2-a\sqrt{-a}\qquad} \\
\qquad\qquad\qquad a\sqrt{-a}+\sqrt{-a} \\
\qquad\qquad\qquad \underline{a\sqrt{-a}+a\qquad\quad} \\
\qquad\qquad\qquad\qquad -a+\sqrt{-a} \\
\qquad\qquad\qquad\qquad -a+\sqrt{-a}
\end{array}
$$

12. The equation can be written,

$$a+y+x\sqrt{c}\cdot\sqrt{-1}=c+x+y\sqrt{a}\cdot\sqrt{-1}.$$

From (**267**) we must have

$$a+y=c+x, \tag{1}$$

and

$$x\sqrt{c}=y\sqrt{a}. \tag{2}$$

Whence, by substituting in (1),

$$x-\frac{\sqrt{c}}{\sqrt{a}}x=a-c$$

$$x=\frac{(a-)c\sqrt{a}}{\sqrt{a}-\sqrt{c}}=a+\sqrt{ac}$$

In the same way,

$$y=c+\sqrt{ac}\qquad \textit{Ans.}$$

(**275**, page 208.)

4. Here we have $a=11$, $\sqrt{b}=6\sqrt{2}$, $b=72$; hence,

$$x=\frac{11+\sqrt{121-72}}{2}=9,$$

$$y=\frac{11-\sqrt{121-72}}{2}=2;$$

$$\sqrt{x}+\sqrt{y}=3+\sqrt{2},\ \textit{Ans.}$$

5. $a=7$, $\sqrt{b}=4\sqrt{3}$, $b=48$; hence,

$$x=\frac{7+\sqrt{49-48}}{2}=4,$$

$$y=\frac{7-\sqrt{49-48}}{2}=3;$$

$$\sqrt{x}-\sqrt{y}=2-\sqrt{3},\ \textit{Ans.}$$

6. $a=7$, $\sqrt{b}=2\sqrt{10}$, $b=40$; hence,

$$x=\frac{7+\sqrt{49-40}}{2}=5,$$

$$y=\frac{7-\sqrt{49-40}}{2}=2;$$

$$\sqrt{x}-\sqrt{y}=\sqrt{5}-\sqrt{2}, \textit{ Ans.}$$

7. $a=94$, $\sqrt{b}=42\sqrt{5}$, $b=8820$; hence,

$$x=\frac{94+\sqrt{8836-8820}}{2}=49,$$

$$y=\frac{94-\sqrt{8836-8820}}{2}=45;$$

$$\sqrt{x}+\sqrt{y}=7+\sqrt{45}=7+3\sqrt{5}, \textit{ Ans.}$$

8. $a=28$, $\sqrt{b}=10\sqrt{3}$, $b=300$; hence,

$$x=\frac{28+\sqrt{784-300}}{2}=55,$$

$$y=\frac{28-\sqrt{784-300}}{2}=3;$$

$$\sqrt{x}+\sqrt{y}=5+\sqrt{3}, \textit{ Ans.}$$

9. $a=np+2m^2$, $\sqrt{b}=2m\sqrt{np+m^2}$, $b=4m^2np+4m^4$; hence,

$$x=\frac{np+2m^2+\sqrt{n^2p^2+4m^2np+4m^4-4m^2np-4m^4}}{2}$$

$$=np+m^2;$$

$$y=\frac{np+2m^2-\sqrt{n^2p^2+4m^2np+4m^4-4m^2np-4m^4}}{2}$$

$$=m^2;$$

$$\sqrt{x}-\sqrt{y}=\sqrt{np+m^2}-m, \textit{ Ans.}$$

10. $a=bc$, $\sqrt{b}=2b\sqrt{bc-b^2}$, $b=4b^2c-4b^4$; hence,

$$x=\frac{bc+\sqrt{b^2c^2-4b^2c+4b^4}}{2}=bc-b^2,$$

$$y=\frac{bc-\sqrt{b^2c^2-4b^2c+4b^4}}{2}=b^2;$$

$$\sqrt{x}+\sqrt{y}=\sqrt{bc-b^2}+b, \textit{ Ans.}$$

11. $a=7$, $\sqrt{b}=30\sqrt{-2}$, $b=-1800$; hence,

$$x=\frac{7+\sqrt{49+1800}}{2}=25,$$

$$y=\frac{7-\sqrt{49+1800}}{2}=-18;$$

$$\sqrt{x}+\sqrt{y}=5+\sqrt{-18}=5+3\sqrt{-2}, \textit{ Ans.}$$

12. $a=16$, $\sqrt{b}=30\sqrt{-1}$, $b=-900$;

$$x=\frac{16+\sqrt{256+900}}{2}=25,$$

$$y=\frac{16-\sqrt{256+900}}{2}=-9;$$

$$\sqrt{x}+\sqrt{y}=5+3\sqrt{-1}$$

also, $$\sqrt{x}-\sqrt{y}=5-3\sqrt{-1}$$

$$10, \quad \textit{Ans.}$$

13. We add the answers of examples 4 and 6, and have

$$3+\sqrt{5}, \textit{ Ans.}$$

14. $a=31$, $\sqrt{b}=12\sqrt{-5}$, $b=-720$; hence,

$$x=\frac{31+\sqrt{961+720}}{2}=36,$$

$$y=\frac{31-\sqrt{961+720}}{2}=-5.$$

Again, $a=-1$, $\sqrt{b}=4\sqrt{-5}$, $b=-80$; hence,

$$x=\frac{-1+\sqrt{1+80}}{2}=4,$$

$$y=\frac{-1-\sqrt{1+80}}{2}=-5.$$

Therefore, $$6+\sqrt{-5}+2+\sqrt{-5}=8+2\sqrt{-5}, \textit{ Ans.}$$

RATIONALIZATION.

(**280**, page 211.)

2. Multiply both terms of the fraction by $\sqrt{6}$.

3. Here the factor is $\sqrt[3]{a^2}$.

4. We have $\dfrac{\sqrt{2}\times\sqrt[3]{81}}{9}=\dfrac{\sqrt{2}\times\sqrt[3]{27\times 3}}{9}=\dfrac{\sqrt[6]{8}\times\sqrt[6]{9}}{3}=\dfrac{\sqrt[6]{72}}{3}$, *Ans.*

5. The factor is $\sqrt{7}-\sqrt{3}$.

6. The factor is $\sqrt{a}+\sqrt{c}$.

7. The factor is $\sqrt{11}+\sqrt{5}$, and we have

$$\frac{11+2\sqrt{55}+5}{6}=\frac{8+\sqrt{55}}{3},\ \textit{Ans.}$$

8. The factor is $\sqrt{11}-\sqrt{3}$.

9. The factor is $\sqrt{10}+\sqrt{6}$; whence,

$$\frac{10+2\sqrt{60}+6}{4}=4+\sqrt{15},\ \textit{Ans.}$$

10. Multiply both terms of the fraction by $\sqrt{5}-\sqrt{-3}$, and we have,

$$\frac{5-2\sqrt{-15}-3}{8}=\frac{1-\sqrt{-15}}{4},\ \textit{Ans.}$$

11. The factor for rationalizing the denominator is $(5+\sqrt{5})(\sqrt{3}-1)$; whence,

$$\frac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)(5+\sqrt{5})(\sqrt{3}-1)}{40}.$$

The product of the first and last factors of the numerator is $2\sqrt{3}$; of the others, $4\sqrt{5}$. Therefore,

$$\frac{8\sqrt{15}}{40}=\tfrac{1}{5}\sqrt{15},\ \textit{Ans.}$$

12. Multiply both numerator and denominator by $1+a+\sqrt{1-a^2}$; whence,

$$\frac{(1+a)^2+2(1+a)\sqrt{1-a^2}+1-a^2}{(1+a)^2-1+a^2}=\frac{2(1+a)+2(1+a)\sqrt{1-a^2}}{2(1+a)a}$$

$$=\frac{1+\sqrt{1-a^2}}{a}, \textit{ Ans.}$$

13. Comparing this with example 2, (**279**), we have $a=5$, $x=2$; and the complete multiplier is

$$(\sqrt{5}+\sqrt[4]{2})(5+\sqrt{2})=5\sqrt{5}+\sqrt{10}+5\sqrt[4]{2}+\sqrt[4]{8}, \textit{ Ans.}$$

14. By (3) the factor is

$$\frac{x^6-y^6}{x-y}=\frac{a^3-b^4}{a^{\frac{1}{2}}-b^{\frac{2}{3}}}=a^{\frac{5}{2}}+a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b^{\frac{4}{3}}+ab^2+a^{\frac{1}{2}}b^{\frac{8}{3}}+b^{\frac{10}{3}}$$

Hence, $$\frac{a^2b^{\frac{2}{3}}+a^{\frac{5}{2}}b^{\frac{4}{3}}+a^2b^2+a^{\frac{3}{2}}b^{\frac{8}{3}}+ab^{\frac{10}{3}}+a^{\frac{1}{2}}b^4}{a^3-b^4}, \textit{ Ans.}$$

RADICAL EQUATIONS.

(**282**, page 214.)

1. Transposing $\sqrt{x}$, and squaring each member of the equation,

$$x+7=49-14\sqrt{x}+x,$$
$$14\sqrt{x}=42,$$
$$\sqrt{x}=3; \qquad x=9, \textit{ Ans.}$$

2. Squaring each member,

$$x^2+6x+9=x^2-4x+59,$$
$$10x=50; \qquad x=5, \textit{ Ans.}$$

3. By squaring, $\sqrt{x+48}=2\sqrt{x}$;
squaring again, $3x=48$; $\qquad x=16$, *Ans.*

4. By cubing, $\sqrt{x+2\sqrt{a+x}}=a-\sqrt{a+x}$.

Squaring this result, $x+2\sqrt{a+x}=a^2-2a\sqrt{a+x}+a+x$,

$$2(1+a)\sqrt{a+x}=(1+a)a,$$

$$\sqrt{a+x}=\frac{a}{2},$$

$$a+x=\frac{a^2}{4}; \qquad x=\frac{a^2-4a}{4}, \textit{ Ans.}$$

5. Multiply by $c\sqrt{x}$, then

$$ac+a=c\sqrt{n}; \text{ or } \qquad x=c(\sqrt{n}-a), \textit{ Ans.}$$

6. Multiply by $\sqrt{1-x^2}$, the least common multiple of the denominators, and we have

$$\sqrt{\frac{(1-x)(1-x)^2}{1-x}}+1+x=3x,$$

or, $$(1-x)+(1+x)=3x; \text{ hence } \qquad x=\tfrac{2}{3}, \textit{ Ans.}$$

7. Multiply by $\sqrt{c+x}$,

$$c+x=\sqrt{a+x^2},$$

squaring, $$c^2+2cx+x^2=a+x; \text{ hence } \qquad x=\frac{a-c^2}{2c}, \textit{ Ans.}$$

8. Transposing x and squaring,

$$9+x\sqrt{x^2-3}=9-6x+x^2,$$

$$\sqrt{x^2-3}=-6+x,$$

$$x^2-3=36-12x+x^2,$$

$$12x=39; \text{ or } \qquad x=3\tfrac{1}{4}, \textit{ Ans.}$$

9. By transposition, $$2\sqrt{x-32}=2(\sqrt{x}-\sqrt{8}),$$

$$x-32=x-2\sqrt{8x}+8,$$

$$\sqrt{8x}=20,$$

$$8x=400; \text{ hence } \qquad x=50, \textit{ Ans.}$$

10. The least common multiple of the denominators is $x-4$, or $(\sqrt{x}-2)(\sqrt{x}+2)$; multiplying by this,

$$a\sqrt{x}+2a-a-c=c\sqrt{x}-2c,$$
$$(a-c)\sqrt{x}=-a+c;$$

therefore, $$x=-\left(\frac{a+c}{a-c}\right)^2, \textit{ Ans.}$$

11. Factoring the first member, we have

$$\frac{\sqrt{m}(\sqrt{x}-1)}{\sqrt{c}(\sqrt{x}-1)}=\frac{\sqrt{x}+m}{\sqrt{x}+c},$$
$$\sqrt{mx}+c\sqrt{m}=\sqrt{cx}+m\sqrt{c},$$
$$\sqrt{x}=\frac{m\sqrt{c}-c\sqrt{m}}{\sqrt{m}-\sqrt{c}}=\sqrt{mc};$$

therefore, $$x=mc, \textit{ Ans.}$$

12. By cubing, $a^3-3a^2x+x^2\sqrt{3a-x}=a^3-3a^2x+3ax^2-x^3$;

$$\sqrt{3a-x}=3a-x,$$

or $$1=\sqrt{3a-x}; \text{ or } x=3a-1, \textit{ Ans.}$$

13. Clearing of fractions,

$$x\sqrt{c^2-ax}+c^2-ax=c^2,$$
$$\sqrt{c^2-ax}=a,$$
$$c^2-ax=a^2; \text{ or } x=\frac{c^2-a^2}{a}, \textit{ Ans.}$$

14. Squaring, $\frac{1}{x^2}+\frac{2}{5}\cdot\frac{1}{x}+\frac{1}{25}=\frac{1}{25}+\frac{1}{x}\sqrt{\frac{1}{5}+\frac{1}{x^2}}$,

transposing, &c., $\frac{1}{x}+\frac{2}{5}=\sqrt{\frac{1}{5}+\frac{1}{x^2}}$,

squaring, $\frac{1}{x^2}+\frac{4}{5}\cdot\frac{1}{x}+\frac{4}{25}=\frac{1}{5}+\frac{1}{x^2}$;

$$\frac{1}{x}=\frac{1}{20}; \text{ hence, } \quad x=20, \textit{ Ans.}$$

15. By squaring, $$a-x=\sqrt{a+x^2}$$

$$a^2-2ax+x^2=a+x^2,$$
$$2ax=a^2-a; \text{ hence, } x=\frac{a-1}{2}, \textit{ Ans.}$$

16. Clearing of fractions,

$$\sqrt[4]{4+5x+x^2}=\sqrt{4-x},$$

$$\sqrt{4+5x+x^2}=4-x,$$

$$4+5x+x^2=16-8x+x^2;\ \text{hence, } x=\tfrac{12}{13},\ \textit{Ans.}$$

17. By transposition,

$$\sqrt[3]{5-x}=\sqrt[3]{10}-\sqrt[3]{5+x},$$

cubing, $5-x=10-\left(3\sqrt[3]{100}\right)\left(\sqrt[3]{5+x}\right)+\left(3\sqrt[3]{10}\right)\left(\sqrt[3]{(5+x)^2}\right)-5-x,$

canceling terms, $\left(\sqrt[3]{10}\right)\left(\sqrt[3]{(5+x)^2}\right)=\left(3\sqrt[3]{100}\right)\left(\sqrt[3]{5+x}\right),$

dividing by $(3\sqrt[3]{10})(\sqrt[3]{5+x})$, $\sqrt[3]{5+x}=\sqrt[3]{10},$

$$5+x=10;\ \text{hence, } x=5,\ \textit{Ans.}$$

NOTE.—If we had transposed the term $\sqrt[3]{5-x}$ and solved, we should have found $x=-5$; and it is evident that either value, $x=\pm 5$, satisfies the original equation.

18. Multiply by $\sqrt{a+x}$,

$$\sqrt{ax+x^2}+a+x=2x,$$

$$\sqrt{ax+x^2}=a-x,$$

$$ax+x^2=a^2-2ax+x^2,$$

therefore, $x=\dfrac{a}{3}$, *Ans.*

19. Squaring, $x^2+2ax+a^2=a^2+\sqrt{b^2x^2+x^4},$

$$x^2+2ax=\sqrt{b^2x^2+x^4},$$

$$x^4+4ax^3+4a^2x^2=b^2x^2+x^4,$$

$$4ax+4a^2=b^2;$$

therefore, $x=\dfrac{b^2-4a^2}{4a}$, *Ans.*

20. Clearing of fractions,

$$24x-8\sqrt{6x}+6\sqrt{6x}-12=24x-9\sqrt{6x}+8\sqrt{6x}-18;$$

whence,

$\sqrt{6x}=6;$ $x=6$, *Ans.*

21. Cubing, we have

$$64+x^2-8x=\frac{(4+x)^3}{4+x};$$

or, $$64+x^2-8x=16+8x+x^2,$$

$$16x=48;\ \text{hence,}\qquad x=3,\ \textit{Ans.}$$

22. Multiplying by $\sqrt{5+x}$,

$$5+x+\sqrt{5x+x^2}=15,$$

$$\sqrt{5x+x^2}=10-x,$$

by squaring, $$5x-x^2=100-20x+x^2,$$

$$25x=100;\ \text{hence,}\qquad x=4,\ \textit{Ans.}$$

23. Multiplying by the denominator $\sqrt{x+\sqrt{x}}$,

$$x+\sqrt{x}-\sqrt{x^2-x}=\tfrac{3}{2}\sqrt{x};$$

dividing by $\sqrt{x}$ and transposing,

$$\sqrt{x-1}=\sqrt{x}-\tfrac{1}{2},$$

squaring, $$x-1=x-\sqrt{x}+\tfrac{1}{4},$$

$$\sqrt{x}=\tfrac{5}{4};\ \text{hence,}\qquad x=\tfrac{25}{16},\ \textit{Ans.}$$

24. Clearing of fractions, we have

$$3ax+5b\sqrt{ax}-3b\sqrt{ax}-5b^2=3ax-2b\sqrt{ax}+3b\sqrt{ax}-2b^2;$$

whence $$\sqrt{ax}=3b,$$

$$ax=9b^2;$$

therefore, $$x=\frac{9b^2}{a},\ \textit{Ans.}$$

25. Clearing of fractions,

$$\sqrt{4x+1}+\sqrt{4x}=9\sqrt{4x+1}-9\sqrt{4x},$$

transposing and dividing by 2,

$$4\sqrt{4x+1}=5\sqrt{4x},$$

Squaring, $$64x+16=100x,$$

$$36x=16;\ \text{hence,}\qquad x=\tfrac{4}{9},\ \textit{Ans.}$$

26. Clearing of fractions,

$$3x+120\sqrt{x}-4\sqrt{x}-160=3x+15\sqrt{x}+6\sqrt{x}+30,$$
$$95\sqrt{x}=190,$$
$$\sqrt{x}=2\text{ ; hence, } x=4, \textit{ Ans.}$$

27. Multiply the numerator and denominator of the left hand member by the numerator,

$$\frac{(\sqrt{x}+\sqrt{x-a})^2}{a}=\frac{n^2a}{x-a},$$

Multiply by a, and take the square root,

$$\sqrt{x}+\sqrt{x-a}=\pm\frac{na}{\sqrt{x-a}},$$

multiplying by $\sqrt{x-a}$,

squaring and transposing, we have, $x=a\dfrac{(1\pm n)^2}{1\pm 2n}$, *Ans.*

PURE QUADRATICS.

(**286**, page 217.)

5. $ax^2+1=(a-x)(a+x),$

By expanding $=a^2-x^2,$

or, $(a+1)x^2=a^2-1,$

$$x^2=a-1\text{ ; hence, } x=\pm\sqrt{a-1}, \textit{ Ans.}$$

6. Clearing of fractions,

$$3(x+4)^2+3(x-4)^2=10(x^2-16),$$

or, $6x^2+96=10x^2-160,$

$$4x^2=256,$$
$$x^2=64\text{ ; hence, } x=\pm 8, \textit{ Ans.}$$

7. As before, $6(x+2)^2+6(x-2)^2=13(x^2-4)$,

or, $$12x^2+48=13x^2-52,$$

$$x^2=100;\ \text{hence, } x=\pm 10,\ \textit{Ans.}$$

8. Clearing of fractions,

$$(x+a)^2+(x-a)^2=7(x^2-a^2)$$

or, $$2x^2+2a^2=7x^2-7a^2,$$

$$5x^2=9a^2,$$

$$x^2=\frac{9a^2}{5};\ \text{hence, } x=\pm\frac{3a}{\sqrt{5}},\ \textit{Ans.}$$

9. Omitting $\frac{x}{4}$ and $\frac{3}{x}$ on each side of the equation it becomes

$$\frac{1}{x}=\frac{x}{12},$$

or, $$x^2=12;\ \text{hence}\qquad x\pm\sqrt{12},\ \textit{Ans.}$$

10. This equation has the same form as example 9; but as we can not unite the letters as we did the numbers in 9, we clear of fractions; whence,

$$cx^2+a^2c=ax^2+ac^2,$$

$$x^2(a-c)=ac(a-c),$$

$$x^2=ac;\ \text{hence}\qquad x=\pm\sqrt{ac},\ \textit{Ans.}$$

11. Clearing of fractions,

$$x^2-8=6+6\sqrt{5},$$

whence, $$x^2=14+6\sqrt{5};$$

and by (**275**), $$x=\pm(3+\sqrt{5}),\ \textit{Ans.}$$

12. Multiply by 6, then

$$2x^2-4\sqrt{2}-3x^2+9=6-6\sqrt{2};$$

whence, $$x^2=3+2\sqrt{2};$$

and by (**275**), $$x=\pm(1+\sqrt{2}),\ \textit{Ans.}$$

13. This equation may be written,

$$x(x+2)=\frac{9(x+2)}{x},$$

whence, $$x^2=9;\ \text{or,}\qquad x=\pm 3,\ \textit{Ans.}$$

It is evident the the value $x=-2$ will satisfy the equation. If we clear the equation of fractions it will be

$$x^3+2x^2-9x-18=0,$$

a complete cubic equation in which x has the three values,

$$x=+3,\quad x=-3,\quad x=-2.$$

14. Clearing of fractions and uniting terms,

$$5x^2=6,$$
$$x^2=1.2\text{; hence}\qquad x=\pm 1.095445+,\ \textit{Ans.}$$

15. Transposing, we have

$$\frac{(x-5)(x+5)}{x+4}=-\tfrac{11}{12}(x-4),$$

or,
$$12(x^2-25)=-11(x^2-16),$$
$$23x^2=476\text{; hence } x=\pm 4.54924+,\ \textit{Ans.}$$

AFFECTED QUADRATICS.

(**220**, page 220.)

The answers to the fourteen first examples can be written down immediately, according to the rule under (**220**).

15. Divide by 3, thus

$$x^2-5x=-4\text{; whence}\qquad x=4 \text{ or } -1,\ \textit{Ans.}$$

16 and 17 are reduced to the general form by division and transposition.

18. Multiplying, we have

$$6x^2-6x-10x+10=2x^2+30,$$
$$4x^2-16x=20,$$

or,
$$x^2-4x=5;\qquad x=5 \text{ or } -1,\ \textit{Ans.}$$

19. This reduces to

$$10x^2-16x+10x-16=5x^2+4x+5x+4,$$
$$5x^2-15x=20,$$
or, $$x^2-3x=4\,;$$
whence, $x=4$, or -1, *Ans.*

20. We have $9x^2+24x+16=54x$
or, $$x^2-\tfrac{10}{3}x=-\tfrac{16}{9},$$
whence, $x=\frac{10}{6}\pm1$; $x=\frac{8}{3}$ or $\frac{2}{3}$, *Ans.*

21. This is already of the general form, and we have

$x=+\frac{1}{15}\pm\frac{23}{30}$; whence $x=\frac{5}{6}$ or $-\frac{7}{10}$, *Ans.*

22. Dividing by 15,

$$x^2+\frac{2x}{45}=\frac{1}{3},$$

$x=-\frac{1}{45}\pm\frac{26}{45}$; whence $x=\frac{5}{9}$ or $-\frac{3}{5}$, *Ans.*

23. Dividing by 4,

$$x^2-\frac{13x}{28}=\frac{5}{72},$$

$x=\frac{13}{56}\pm\frac{59}{168}$; whence $x=\frac{7}{12}$ or $-\frac{5}{42}$, *Ans.*

24. Multiplying by $4(x^2-1)$, the least common multiple of the denominators, we have

$$2(x+1)+12=x^2-1,$$
or, $x^2-2x=15$; whence $x=5$ or -3, *Ans.*

25. Clearing of fractions,

$$4(x+2)(x+3)+5(x+1)(x+3)=12(x+1)(x+2)\,;$$
or,
$$4x^2+12x+8x+24+5x^2+15x+5x+15=12x^2+24x+12x+24.$$
$$3x^2-4x=15,$$
$$x^2-\tfrac{4}{3}x=5\,;$$
whence, $x=3$ or $-\frac{5}{3}$, *Ans.*

(291, page 223.)

1. Multiplying by 5, and adding 2^2, the complete equation is

$$25x^2+20x+4=1024,$$
$$5x+2=\pm 32; \qquad x=6 \text{ or } \tfrac{34}{5}, \textit{ Ans.}$$

2. Multiplying by 5, and adding 2^2,

$$25x^2+20x+4=1369,$$
$$5x+2=\pm 37; \qquad x=7 \text{ or } -\tfrac{39}{5}, \textit{ Ans.}$$

3. Multiplying by 7, and adding 10^2,

$$49x^2-140x+100=324,$$
$$7x-10=\pm 18; \qquad x=7 \text{ or } -\tfrac{8}{7}, \textit{ Ans.}$$

4. Multiplying by 24, and adding 15^2,

$$144x^2+360x+225=441,$$
$$12x+15=\pm 21; \qquad x=\tfrac{1}{2} \text{ or } -3, \textit{ Ans.}$$

5. Multiplying by 8, and adding 5^2.

$$16x^2-40x+25=961,$$
$$4x-5=\pm 31; \qquad x=9 \text{ or } -\tfrac{13}{2}, \textit{ Ans.}$$

6. Multiplying by 21, and adding 146^2,

$$441x^2-292\times 21x+21316=10816,$$
$$21x-146=\pm 104; \qquad x=11\tfrac{19}{21} \text{ or } 2, \textit{ Ans.}$$

7. Multiplying by 24, and adding 13^2,

$$144x^2-13\times 24x+169=25,$$
$$12x-13=\pm 5; \qquad x=\tfrac{3}{2} \text{ or } \tfrac{2}{3}, \textit{ Ans.}$$

8. Multiplying by 28, and adding 3^2.

$$196x^2-84x+9=4489,$$
$$14x-3=\pm 67; \qquad x=5 \text{ or } -\tfrac{32}{7}, \textit{ Ans.}$$

9. Multiplying by 12, and adding 53^2,

$$36x^2-53\times 36x+2809=2401,$$
$$6x-53=\pm 49; \qquad x=17 \text{ or } \tfrac{2}{3}, \textit{ Ans.}$$

10. Multiply by 4, and add 13^2,

$$4x^2+52x+169=729,$$
$$2x+13=\pm 27; \qquad x=7 \text{ or } -20, \textit{ Ans.}$$

11. Multiply by 3, and add 4^2,

$$9x^2-24x+16=31+12\sqrt{3},$$
$$3x-4=\pm(2+3\sqrt{3}),$$

whence, $\qquad x=2+\sqrt{3}$ or $\frac{2}{3}-\sqrt{3}$, *Ans.*

12. Multiply by 4, and add 11^2,

$$4x^2+44x+121=441,$$
$$2x+11=\pm 21; \qquad x=5 \text{ or } -16, \textit{ Ans.}$$

13. Clearing of fractions and uniting terms,

$$21x^2-292x=-500,$$

which is example 6.

14. Clearing of fractions and uniting terms,

$$x^2-27x=-252,$$
$$4x^2-148x+1369=361,$$
$$2x-37=\pm 19; \qquad x=28 \text{ or } 9, \textit{ Ans.}$$

15. Clearing of fractions and uniting terms,

$$2x^2-13x=-6,$$
$$16x^2-104x+169=121,$$
$$4x-13=\pm 11; \qquad x=6 \text{ or } \tfrac{1}{2}, \textit{ Ans.}$$

16. Clearing of fractions and uniting terms,

$$4x^2-11x=-7,$$
$$64x^2-176x+121=9,$$
$$8x-11=\pm 3; \qquad x=\tfrac{7}{4} \text{ or } 1, \textit{ Ans.}$$

17. Clearing of fractions and uniting terms,

$$21x^2-94x=-13,$$
$$441x^2-94\times 21x+2209=1936,$$
$$21x-47=\pm 44; \qquad x=\tfrac{13}{3} \text{ or } \tfrac{1}{7}, \textit{ Ans.}$$

(223—224)

(**293**, page 225.)

1. Here we have $t=\frac{3}{2}$, and

$$16x^2+12x+\tfrac{9}{4}=\tfrac{49}{4},$$
$$4x+\tfrac{3}{2}=\pm\tfrac{7}{2};\qquad x=\tfrac{1}{2}\text{ or }-\tfrac{5}{4},\ \textit{Ans.}$$

2. $t=\frac{5}{12}$;

$$36x^2-5x+\tfrac{25}{144}=\tfrac{64}{144},$$
$$6x-\tfrac{5}{12}=\pm\tfrac{8}{12};$$

whence, $x=\frac{13}{72}$ or $-\frac{1}{24}$, *Ans.*

3. $t=\frac{2}{3}$;

$$81x^2-12x+\tfrac{4}{9}=\tfrac{1}{9},$$
$$9x-\tfrac{2}{3}=\pm\tfrac{1}{3};$$

whence, $x=\frac{1}{9}$ or $\frac{1}{27}$, *Ans.*

4. $t=\frac{3}{7}$;

$$\frac{49}{25}x^2+\frac{6x}{5}+\frac{9}{49}=1,$$
$$\tfrac{7}{5}x+\tfrac{3}{7}=\pm 1;$$

whence, $x=\frac{20}{49}$ or $-\frac{50}{49}$, *Ans.*

5. $t=5$;

$$\tfrac{841}{625}x^2-\tfrac{58}{5}x+25=36,$$
$$\tfrac{29}{25}x-5=\pm 6,$$

whence, $x=\frac{275}{29}$ or $-\frac{25}{29}$, *Ans.*

6. Clearing of fractions and uniting terms,

$$49x^2-80x=36.$$

Hence $t=\frac{40}{7}$,

$$49x^2-80x+\tfrac{1600}{49}=\tfrac{3364}{49},$$
$$7x-\tfrac{40}{7}=\pm\tfrac{58}{7};$$

whence, $x=2$ or $-\frac{18}{49}$, *Ans.*

(**294**, page 226.)

1. Put $2a=7$, and we have

$$x^2-20x=2a+1,$$
$$x^2-2ax+a^2=a^2+2a+1,$$
$$x-a=\pm(a+1),$$
$$x=2a+1,\text{ or }-1;$$

whence, $x=8$ or -1, *Ans.*

2. Put $2a=11$, and add a^2;

$$x^2+2ax+a^2=a^2+4a+4,$$
$$x+a=\pm(a+2),$$
$$x=-(2a+2) \text{ or } +2;$$

whence, $x=-13$ or 2, *Ans.*

3. Put $2a=17$, and add a^2;

$$x^2-2a+a^2=a^2+6a+9,$$
$$x-a=\pm(a+3),$$
$$x=2a+3 \text{ or } -3;$$

whence, $x=20$ or -3, *Ans.*

4. Put $2a=21$, and add a^2;

$$x^2+2a+a^2=a^2+4a+4,$$
$$x+a=\pm(a+2);$$

whence, $x=2$ or -23, *Ans.*

5. Put $2a=75$, and add a^2;

$$x^2-2ax+a^2=a^2+2a+1,$$
$$x-a=\pm(a+1);$$

whence, $x=76$ or -1, *Ans.*

6. Put $2a=72$, and add a^2;

$$x^2+2ax+a^2=a^2+10a+25,$$
$$x+a=\pm(a+5);$$

whence, $x=5$ or -77, *Ans.*

7. Put $2a=325$, and add a^2;

$$x^2-2ax+a^2=a^2+20a+100,$$
$$x-a=\pm(a+10);$$

whence, $x=335$ or -10, *Ans.*

EQUATIONS IN THE QUADRATIC FORM.

(296, page 232.)

In solving these examples, we shall put y equal to the lowest power of the unknown quantity.

1. Put $y=x^2$;

$$y^2-34y=-225,$$
$$y^2-34y+289=64,$$
$$y=17\pm 8,$$
$$x^2=y=25, \text{ or } 9;$$

whence, $x=\pm 5$ or ± 3, *Ans.*

2. $y=x^3$;

$$y^2-35y=-216,$$
$$4y^2-140y+1225=361,$$
$$2y-35=\pm 19,$$
$$x^3=y=27 \text{ or } 8;$$

whence, $x=3$ or 2, *Ans.*

3. $y=x^3$;

$$y^2-4y+4=625,$$
$$y-2=\pm 25,$$
$$x^3=y=27 \text{ or } -23;$$

whence, $x=3$ or $\sqrt[3]{-23}$, *Ans.*

4. $y=x^5$;

$$y^2+31y=32$$
$$4y^2+31y+961=1089,$$
$$2y+31=\pm 33,$$
$$x^5=y=1 \text{ or } 32;$$

whence, $x=1$ or 2, *Ans.*

5. $y=x^{\frac{3}{2}}$;

$$y^2-y=56,$$
$$4y^2-4y+1=225,$$
$$2y-1=\pm 15,$$
$$x^{\frac{3}{2}}=y=8 \text{ or } -17,$$
$$x^{\frac{1}{2}}=2 \text{ or } \sqrt[3]{-7};$$

whence, $x=4$ or $\sqrt[3]{49}$, *Ans.*

6. $y=x^n$;

$$y^2-2y+1=9,$$
$$y-1=\pm 3,$$
$$x^n=y=4 \text{ or } -2;$$

whence, $x=\sqrt[n]{4}$ or $\sqrt[n]{-2}$, *Ans.*

7. $y=x^{\frac{1}{n}}$;

$$20y^2-31y=-12,$$
$$1600y^2-2480y+961=\ \ 1,$$
$$40y-31=\pm 1,$$
$$x^{\frac{1}{n}}=y=\tfrac{4}{5} \text{ or } -\tfrac{3}{4};$$

whence, $x=(\frac{4}{5})^n$, or $(\frac{3}{4})^n$, *Ans.*

8. $y=\sqrt[3]{x}$;

$$3y^2-10y=-3,$$
$$36y^2-120y+100=\ \ 64,$$
$$\sqrt[3]{x}=y=\ \ 3 \text{ or } \tfrac{1}{3};$$

whence, $x=27$ or $\frac{1}{27}$, *Ans.*

9. $y=\sqrt{x+5}$;

$$y^2-y=6,$$
$$4y^2-4y+1=25,$$
$$\sqrt{x+5}=y=3 \text{ or } -2;$$

whence, $x=4$ or -1, *Ans.*

10. $y=(x+12)^{\frac{1}{4}}$;

$$y^2+y=\ \ 6,$$
$$4y^2+4y+1=25,$$
$$(x+12)^{\frac{1}{4}}=y=\ \ 2 \text{ or } -3,$$
$$x+12=16 \text{ or } 81;$$

whence, $x=4$ or 69, *Ans.*

11. $y=(x+a)^{\frac{1}{4}}$;

$$y^2+2by=3b^2,$$
$$(x+a)^{\frac{1}{4}}=y=b \text{ or } -3b;$$

whence, $x=b^4-a$, or $81b^4-a$, *Ans.*

12. Transposing x and squaring,

$$5x+10=x^2-16x+64,$$

or $$x^2-21x=-54,$$

which gives $$x=18 \text{ or } 3, \textit{ Ans.}$$

13. $y=\sqrt{9x+4}$;

$$y^2+2y+1=16,$$

$$\sqrt{9x+4}=y=3 \text{ or } -5,$$

$$9x+4=9 \text{ or } 25; \qquad x=\tfrac{5}{9} \text{ or } \tfrac{7}{3}, \textit{ Ans.}$$

14. $y=\sqrt[4]{10+x}$;

$$y^2-y=\ 2,$$

$$4y^2-4y-1=\ 9,$$

$$\sqrt[4]{10+x}=y=\ 2 \text{ or } -1,$$

$$10+x \quad =16 \text{ or } \quad 1;$$

whence, $$x=6 \text{ or } -9, \textit{ Ans.}$$

15. $y=(x-5)^{\frac{3}{2}}$;

$$y^2-3y+\tfrac{9}{4}=\tfrac{169}{4},$$

$$(x-5)^{\frac{3}{2}}=y=8 \text{ or } -5,$$

$$x-5 \quad =4 \text{ or } \sqrt[3]{(-5)^2};$$

whence, $$x=9, \text{ or } 5+\sqrt[3]{(-5)^2}, \textit{ Ans.}$$

16. $y=(1+x-x^2)^{\frac{1}{2}}$;

$$2y^2-y=-\tfrac{1}{9},$$

$$16y^2-8y+1=\ \tfrac{1}{9},$$

whence, $$y=\ \tfrac{1}{3} \text{ or } -\tfrac{1}{6}.$$

Hence, $1+x-x^2=\frac{1}{9}$, or $\qquad 1+x-x^2=\frac{1}{36}$,

$$x^2-x+\tfrac{1}{4}=\tfrac{41}{36}, \qquad x^2-x+\tfrac{1}{4}=\tfrac{11}{9},$$

$$x-\tfrac{1}{2}=\pm\tfrac{1}{6}\sqrt{41}, \qquad x-\tfrac{1}{2}=\pm\tfrac{1}{3}\sqrt{11},$$

whence, $$x=\tfrac{1}{2}\pm\tfrac{1}{6}\sqrt{41}, \qquad x=\tfrac{1}{2}\pm\tfrac{1}{3}\sqrt{11}, \textit{ Ans.}$$

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17. $y=\sqrt{x+16}$;

$$y^2-3y+\tfrac{9}{4}=\tfrac{49}{4},$$
$$y=\tfrac{3}{2}\pm\tfrac{7}{2},$$
$$\sqrt{x+16}=5 \text{ or } -2;$$

whence, $x=9$ or -12, *Ans.*

18. Clearing of fractions and uniting terms,

$$81x^4-82x^2=-1,$$
$$81x^4-82x^2+\tfrac{1681}{81}=\tfrac{1600}{81},$$
$$9x^2-\tfrac{41}{9}=\pm\tfrac{40}{9};$$

whence, $x=\pm1$ or $\pm\frac{1}{9}$, *Ans.*

If we add 1 to each side of this equation, we shall have perfect squares; thus,

$$81x^2+18+\frac{1}{x^2}=100,$$

and $$9x+\frac{1}{x}=\pm10;$$

hence two quadratics for x, and the same values as before.

19. Clearing of fractions and uniting terms,

$$225x^4-901x^2=-4,$$
$$225x^4-901x^2+\tfrac{811801}{900}=\tfrac{808201}{900},$$
$$15x^2=\tfrac{901}{30}\pm\tfrac{899}{30},$$
$$x^2=4 \text{ or } \tfrac{1}{225};$$

whence, $x=\pm2$ or $\pm\frac{1}{15}$, *Ans.*

Here, if we add $\frac{4}{3}$ to each side, we have

$$25x^2+\frac{20}{3}+\frac{4}{9x^2}=\frac{961}{9},$$

$$5x+\frac{2}{3x}=\pm\frac{31}{3};$$

whence two quadratics for x.

20.
$$\begin{array}{lll}
 & x^4+2x^3-7x^2-8x+12 & (x^2+x \\
 & x^4 & \\
2x^2+x & \quad +2x^3-7x^2 & \\
 & \quad\ \ 2x^3+\ x^2 & \\
 & \qquad\ -8x^2-8x, &
\end{array}$$

Hence, by taking the square root thus far we see that the expression may be written,

$$(x^2+x)^2-8(x^2+x)=-12;$$

or if $y=x^2+x$,
$$y^2-8y=-12,$$
$$y^2-8y+16=\ \ 4,$$
$$y-4=\pm 2.$$

Hence,
$$x^2+x-4=\ \ 2, \text{ or } \qquad x^2+x-4=-2,$$
$$x^2+x+\tfrac{1}{4}=\ \tfrac{25}{4}, \qquad x^2+x+\tfrac{1}{4}=\ \tfrac{9}{4},$$
$$x+\tfrac{1}{2}=\pm\tfrac{5}{2}, \qquad x+\tfrac{1}{2}=\pm\tfrac{3}{2},$$

and $\quad x=2$ or -3, $\qquad x=1$ or -2, *Ans.*

21. Multiplying by x, the equation is

$$\begin{array}{lll}
 & x^4-8x^3+19x^2-12x=0, & (x^2-4x \\
 & x^4 & \\
2x^2-4x & \ -8x^3 & \\
 & \ -8x^3+16x^2 & \\
 & \qquad\quad 3x^2-12x, &
\end{array}$$

Hence,
$$(x^2-4x)^2+3(x^2-4x)\quad =0,$$
$$(x^2-4x)^2+3(x^2-4x)+\tfrac{9}{4}=\tfrac{9}{4},$$
$$x^2-4x=0 \text{ or } -3;$$
$$x^2-4x+4=4, \qquad x^2-4x+4=1,$$

and $\quad x=0$ or 4, $\qquad x=1$ or 3, *Ans.*

22.
$$\begin{array}{lll}
 & x^4-10x^3+35x^2-50x+24=0, & (x^2-5x \\
 & x^4 & \\
2x^2-5x & -10x^3+35x^2 & \\
 & -10x^3+25x^2 & \\
 & \qquad\quad 10x^2-50x &
\end{array}$$

Hence,

$$(x^2-5x)^2+10(x^2-5x) \quad =-24,$$
$$(x^2-5x)^2+10(x^2-5x)+25= \quad 1,$$
$$x^2-5x=-4 \text{ or } -6;$$
$$x^2-5x+\tfrac{25}{4}=\tfrac{9}{4}, \qquad x^2-54+\tfrac{25}{4}= \quad \tfrac{1}{4},$$

and $x=1$ or 4, $\qquad x=2$ or 3, *Ans.*

23. $x^4-8ax^3+\ 8a^2x^2+32a^3x-9a^4=0, \ (x^2-4ax$

$$\begin{array}{ll} & x^4 \\ \hline 2x^2-4ax & -8ax^3+\ 8a^2x^2 \\ & -8ax^3+16a^2a^2 \\ \hline & -\ 8a^2x^2+32a^3x. \end{array}$$

Hence, $\quad (x^2-4ax)^2-8a^2(x^2-4ax) \quad =9a^4$

$$(x^2-4ax)^2-8a^2(x^2-4ax)+16a^4=25a^4$$
$$x^2-4ax \qquad =9a^2 \text{ or } -a^2$$
$$x^2-4ax+4a^2=13a^2 \quad \text{or } x^2-4ax+4a^2=5a^2;$$

and $\qquad x=a\left(2\pm\sqrt{13}\right)$ or $x=a(2\pm\sqrt{3})$, *Ans.*

24. Performing the multiplication indicated,

$$y^4-2cy^3+c^2y^2-2y^2+2cy=c^2,$$

or, $\qquad (y^2-cy)^2-2(y^2-cy)=c^2,$

$$y^2-cy=1\pm\sqrt{1+c^2};$$

whence $\qquad y=\frac{c}{2}\pm\left(\frac{c^2}{4}+1\pm\sqrt{1+c^2}\right)^{\frac{1}{2}}$, *Ans.*

PROMISCUOUS EXAMPLES IN QUADRATICS.

1. Completing the square, we have

$$4x^2+44x+121=441,$$
$$2x+11=\pm 21;$$

whence, $\qquad x=5$ or -16, *Ans.*

2. Clearing of fractions and uniting terms,

$$3x^2-9x=12$$
$$x^2-3x+\tfrac{9}{4}=\tfrac{25}{4},$$
$$x-\tfrac{3}{2}=\pm\tfrac{5}{2};$$

whence, $x=4$ or -1, *Ans.*

3. Clearing of fractions and uniting terms,

$$x^2+x=6,$$
$$x^2+x+\tfrac{1}{4}=\tfrac{25}{4},$$
$$x+\tfrac{1}{2}=\tfrac{5}{2};$$

whence, $x=2$ or -3, *Ans.*

4. Multiplying by $8(x^2-1)$, the least common multiple of the denominators, we have

$$12-2x+2=x^2-1,$$

or, $$x^2+2x+1=16;$$

whence, $x=3$ or -5, *Ans.*

5. Clearing of fractions,

$$(2x^2+x-5)(x+19)=(x^2+4x+3)(2x+15),$$

or, $$16x^2-52x=140;$$
$$16x^2-52x+\tfrac{169}{4}=\tfrac{729}{4},$$
$$4x=\tfrac{13}{2}\pm\tfrac{27}{2};$$

whence, $x=5$ or $-\tfrac{7}{4}$, *Ans.*

6. Clearing of fractions,

$$x^3-10x^2+1=\quad(x-3)^3,$$

or, $$x^2+27x=\quad 28,$$
$$x^2+27x+\tfrac{729}{4}=\quad\tfrac{841}{4},$$
$$x+\tfrac{27}{2}=\pm\tfrac{29}{2};$$

whence, $x=1$ or -28, *Ans.*

Ex. 7 and 8 need no solution here.

9. Add mn to both sides, (**293**),

$$mx^2-2mx\sqrt{n}+mn=\quad nx^2,$$
$$\sqrt{m}\cdot x-\sqrt{mn}=\pm\sqrt{n}\cdot x;$$

whence, $x=\dfrac{\sqrt{mn}}{\sqrt{m}\pm\sqrt{n}}$, *Ans.*

10. Add $\frac{4}{9}$ to each side, (**293**);

$$\frac{4x^2}{49}+\frac{8x}{21}+\frac{4}{9}=\frac{64}{9},$$

$$\tfrac{2}{7}x+\tfrac{2}{3}=\pm\tfrac{8}{3};$$

whence, $x=7$ or $-11\frac{2}{3}$, *Ans.*

11. Add 36 to each side, (**293**),

$$\frac{x^2}{361}-\frac{12x}{19}+36=4,$$

$$\frac{x}{19}-6=\pm 2;$$

whence, $x=152$ or 76, *Ans.*

12. Transposing, we have

$$\frac{16}{(2x-4)^4}-\frac{8}{(2x-4)^2}+1=0,$$

$$\frac{4}{(2x-4)^2}-1=0,$$

$$(2x-4)^2=4,$$

$$2x-4=\pm 2;$$

whence, $x=3$ or 1, *Ans.*

13. Put $y=\sqrt{x^2+11}$;

$$y^2+y=42,$$

$$y^2+y+\tfrac{1}{4}=\tfrac{169}{4},$$

$$y=6 \text{ or } -7,$$

$$x^2+11=36 \text{ or } 49;$$

whence, $x=\pm 5$, or $\pm\sqrt{38}$, *Ans.*

14. Add 5 to each side of the equation, and put $y=\sqrt{x^2-2x+5}$; then,

$$y^2+6y=16,$$

$$y^2+6y+9=25,$$

$$y=2 \text{ or } -8;$$

$$x^2-2x+5=4, \text{ or } \quad x^2-2x+5=64,$$

$$x^2-2x+1=0, \quad x^2-2x+1=60,$$

whence, $x=1$, $x=1\pm 2\sqrt{15}$, *Ans.*

15. Add $\frac{289}{16}x^2$ to each side, (293).

$$x^4+\tfrac{17}{2}x^3+\tfrac{289}{16}x^2=\tfrac{289}{16}x^2+34x+16,$$
$$x^2+\tfrac{17}{4}x=\pm(\tfrac{17}{4}x+4)$$

Hence,

$$x^2=4, \text{ or } \qquad x^2+\tfrac{17}{2}x=-4,$$

and $x=\pm 2$, $\qquad x=-8$, or $-\frac{1}{2}$, *Ans.*

16. This equation may be written,

$$x-1=\frac{2(\sqrt{x}+1)}{\sqrt{x}};$$

dividing by $\sqrt{x}+1$, we have

$$\sqrt{x}-1=\frac{2}{\sqrt{x}},$$
$$x-\sqrt{x}=2,$$
$$x-\sqrt{x}+\tfrac{1}{4}=\tfrac{9}{4},$$
$$\sqrt{x}+\tfrac{1}{2}=\pm\tfrac{3}{2};$$

whence, $\qquad x=1$ or 4, *Ans.*

17. Clearing of fractions,

$$2x+2\sqrt{x}=16-x,$$
$$3x+2\sqrt{x}=16,$$
$$36x+24\sqrt{x}+4=196,$$
$$6\sqrt{x}+2=\pm 14;$$

whence, $\qquad x=4$ or $7\frac{1}{9}$, *Ans.*

18. By squaring,

$$x^2\left(3\sqrt{2}-\frac{x^2}{2\sqrt{2}}\right)=\frac{(2+x^2)^2}{\sqrt{8}},$$

or,

$$\frac{x^2(12-x^2)}{2\sqrt{2}}=\frac{(2+x^2)^2}{2\sqrt{2}}.$$

Hence,

$$x^4-4x^2=-2,$$
$$x^2=2\pm\sqrt{2};$$

and $\qquad x=\pm(2\pm\sqrt{2})^{\frac{1}{2}}$, *Ans.*

19. This equation may be written,

$$\sqrt{(x+1)\cdot\frac{x-1}{x}}-\sqrt{\frac{x-1}{x}}=\frac{x-1}{x},$$

which is divisible by $\sqrt{\frac{x-1}{x}}$; whence,

$$\sqrt{x-1}-1=\sqrt{\frac{x-1}{x}}.$$

By squaring, $x+1-2\sqrt{x+1}+1=\frac{x+1}{x}$,

or, $2\sqrt{x+1} \quad =\frac{x^2+x+1}{x}$;

squaring again, and clearing of fractions,

$$4x^3+4x^2=x^4+2x^3+3x^2+2x+1,$$

or, $x^4-2x^3-x^2+2x+1=0$;

taking square root, $x^2-x-1=0$,

$$x-\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{5};$$

whence, $x=\frac{1}{2}(1\pm\sqrt{5})$, *Ans.*

20. Multiply both terms of the first fraction by $1+\sqrt{1-x^2}$, of the second by $1-\sqrt{1-x^2}$; then

$$\frac{1+\sqrt{1-x^2}}{x^2}-\frac{1-\sqrt{1-x^2}}{x^2}=\frac{\sqrt{3}}{x^2},$$

or, $2\sqrt{1-x^2}=\sqrt{3}$,

squaring, $x^2=\frac{1}{4}$;

whence, $x=\pm\frac{1}{2}$, *Ans.*

21. This may be written,

$$\left(\left(\frac{1}{1+x}\right)^{\frac{3}{2}}\right)^{\frac{1}{3}}=\frac{\sqrt{2x}}{12},$$

$$\left(\frac{1}{1+x}\right)^{\frac{1}{2}}=\frac{\sqrt{2x}}{12}$$

$$\frac{1}{1+x}=\frac{2x}{144},$$

$$x^2+x=72,$$

$$4x^2+4x+1=289,$$

whence, $2x+1=\pm17$; $x=8$ or -9, *Ans.*

22. Multiply both terms of the fraction by $x+\sqrt{x^2-9}$, and take the indicated square root; then

$$\frac{x+\sqrt{x^2-9}}{3}=x-2,$$

$$\sqrt{x^2-9}=2x-6,$$

$$x^2-9=4x^2-24x+36\,;$$

whence,
$$x^2-8x=-15,$$

$$x^2-8x+16=1\,;$$

and
$$x=5 \text{ or } 3, \textit{ Ans.}$$

23. Completing the square, we have

$$x^{\frac{6}{5}}+x^{\frac{3}{5}}+\tfrac{1}{4}=\tfrac{3025}{4},$$

$$x^{\frac{3}{5}}+\tfrac{1}{2}=\pm\tfrac{55}{2},$$

$$x^{\frac{3}{5}}\quad=27 \text{ or } -28,$$

$$x^{\frac{1}{5}}\quad=3 \text{ or } \sqrt[3]{-28}\,;$$

whence,
$$x=243 \text{ or } \sqrt[3]{(-28)^5}, \textit{ Ans.}$$

Since the exponents are odd, we may write the last value of x thus,

$$-(28)^{\frac{5}{3}}=-(2\times2\times7)^{\frac{5}{3}}=-(32\times32\times7^5)^{\frac{1}{3}}$$

$$=-(512\times2\times7^5)^{\frac{1}{3}}=-8(2\times7^5)^{\frac{1}{3}}$$

$$=-8(33614)^{\frac{1}{3}}.$$

24. Completing the square

$$144x^{2n}-312x^n+169=25,$$

$$12x^n-13=\pm5\,;$$

whence,
$$x=\sqrt[n]{\tfrac{3}{2}} \text{ or } \sqrt[n]{\tfrac{2}{3}}, \textit{ Ans.}$$

25. Transposing $2x$ and squaring, we have

$$2+2x=c^2-2c^2x-4cx+c^2x^2+4cx^2+4x^2,$$

or,
$$(c+2)^2x^2-2(c+1)^2x=2-c^2\,;$$

by (**293**),
$$(c+2)^2x^2-2(c+1)^2x+\frac{(c+1)^4}{(c+2)^2}=\frac{(2c+3)^2}{(c+2)^2},$$

$$(c+2)x-\frac{(c+1)^2}{c+2}=\pm\frac{2c+3}{c+2}\,;$$

whence,
$$x=1 \text{ or } \frac{c^2-2}{(c+2)^2}, \textit{ Ans.}$$

26. By expanding,

$$(a+x)^5 = \quad x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5,$$
$$-(a-x)^5 = -x^5+5x^4a-10x^3a^2+10x^2a^3-5xa^4+a^5.$$

Therefore we have

$$10ax^4+20a^3x^2+2a^5=352a^5,$$
$$x^4+\;2a^2x^2=\;35a^4,$$
$$x^4+\;2a^2x^2+\;a^4=\;36a^4,$$
$$x^2+\;a^2=\pm 6a^2;$$

whence, $x=\pm a\sqrt{5}$, or $\pm a\sqrt{-7}$, *Ans.*

27. Clearing of fractions and uniting terms, we have

$$a^2cx^2-a(b+2c)x=-(b+2c);$$

by (**293**), $$a^2cx^2-a(b+2c)x+\frac{(b+2c)^2}{4c}=\frac{b^2-4c^2}{4c},$$

$$a\sqrt{c}x-\frac{b+2c}{2\sqrt{c}}=\pm\frac{1}{2\sqrt{c}}\sqrt{b^2-4c^2},$$

whence, $$x=\frac{1}{2ac}(b+2c\pm\sqrt{b^2-4c^2}),\ \textit{Ans.}$$

28. Clearing of fractions,

$$abx=(a+b+x)bx+(a+b+x)ax+(a+b+x)ab;$$

multiplying and arranging terms,

$$(a+b)x^2+(a+b)^2x=-ab(a+b),$$
$$x^2+(a+b)x=-ab,$$
$$x=-\frac{a+b}{2}\pm\frac{a-b}{2};$$

whence, $x=-a$ or $-b$, *Ans.*

29. Clearing of fractions and transposing, we have

$$x^4+8x^3+16x^2-a^2x^2-8a^2x-16a^2=0.$$

Extracting the square root according to (**296**), this equation may be written,

$$(x^2+4x)^2-a^2(x+4)^2=0,$$

or, $$(x^2+4x)^2=a^2(x+4)^2,$$
$$x^2+4x=\pm a(x+4).$$

Dividing by $x+4$ $\qquad x=\pm a$, *Ans.*

30. Multiply both terms of the left hand member by the denominator;

$$\frac{(\sqrt{x+a}-\sqrt{x-a})^2}{2a}=\frac{x}{2a}.$$

Omitting the factor $2a$, and squaring the left hand member,

$$2x-2\sqrt{x^2-a^2}=x,$$
$$4x^2-4a^2=x^2,$$
$$x^2=\frac{4a^2}{3};$$

whence, $x=\pm 2a\sqrt{\tfrac{1}{3}}$, *Ans.*

31. Dividing out the factor $a-x$ in the second term, and clearing of fractions, we have

$$a^2+x^2+a^2+2ax+x^2=4a^2+4ax,$$

or, $$x^2-ax=a^2,$$

$$x^2-ax+\frac{a^2}{4}=\frac{5}{4}a^2;$$

whence, $x=\frac{a}{2}(1\pm\sqrt{5})$, *Ans.*

32. Clearing of fractions and transposing, we have

$$\sqrt{2ax+x^2}=ab+bx-a-x;$$

squaring and uniting terms,

$$(2b-b^2)x^2+2a(2b-b^2)x=a^2+a^2b^2-2a^2b,$$
$$x^2+2ax=\frac{a^2+a^2b^2-2a^2b}{2b-b^2},$$
$$x^2+2ax+a^2=\frac{a^2}{2b-b^2},$$
$$x=-a\pm\frac{a}{\sqrt{2b-b^2}};$$

whence, $x=\pm a\left\{\frac{1\mp\sqrt{2b-b^2}}{\sqrt{2b-b^2}}\right\}$ *Ans.*

33. Squaring the left hand member as indicated,

$$\frac{4+4x+x^2}{4-4x+x^2}=\frac{2b+2x}{2b};$$

clearing of fractions and uniting terms,

$$cx^3-4cx^2+4cx=16bx,$$

or,

$$x^2-4x+4=\frac{16b}{c},$$

$$x-2=\pm 4\sqrt{\frac{b}{c}};$$

whence,

$$x=2\left(1\pm 2\sqrt{\frac{b}{c}}\right), \textit{ Ans.}$$

NOTE.—The value $x=0$, satisfies the original equation, and also our reduced equation after uniting terms.

EXAMPLES OF SIMULTANEOUS EQUATIONS.

(**301**, page 243.)

1. From the second equation, we have $x=2y^2$, and by substitution in the first,

$$\begin{aligned} 2y^2-y&=15, \\ 16y^2-8y+1&=121, \\ 4y&=12 \text{ or } -10, \\ y&=\ 3 \text{ or } -\ 2\tfrac{1}{2} \\ 2y^2=x&=18 \text{ or } \quad 12\tfrac{1}{2} \end{aligned} \Bigg\} \textit{ Ans.}$$

2. The second equation gives by transposing y and multiplying by $\frac{y}{2}$,

$$xy=11y-\frac{y^2}{2};$$

and substituting in the first equation,

$$\begin{aligned} 3y^2+22y&=240, \\ 9y^2+66y+121&=841, \\ 3y&=-11\pm 29; \\ y&=\ \ 6 \text{ or } -13\tfrac{1}{3} \\ 2x=22-y; \qquad x&=\ \ 8 \text{ or } \quad 17\tfrac{2}{3} \end{aligned} \Bigg\} \textit{ Ans.}$$

3. By substitution from the second equation, and reducing,

$$4y^2+9y=100,$$
$$4y^2+9y+\tfrac{81}{16}=\tfrac{1681}{16},$$
$$2y+\tfrac{9}{4}=\pm\tfrac{41}{4},$$
$$\left.\begin{aligned} y&=4 \text{ or } -6\tfrac{1}{4} \\ x&=9 \text{ or } -14\tfrac{1}{16} \end{aligned}\right\} Ans.$$
$$x=\frac{9y}{4};$$

4. By adding the equations, and dividing by 5,

$$x^2+x=12,$$
$$x^2+x+\tfrac{1}{4}=\tfrac{49}{4},$$
$$x=-\tfrac{1}{2}\pm\tfrac{7}{2},$$
$$y=25-5x; \qquad \left.\begin{aligned} x&=\ \ 3 \text{ or } -\ 4 \\ y&=10 \text{ or } -45 \end{aligned}\right\} Ans.$$

5. Multiplying the second equation by 3, and adding,

$$13x^2=52,$$
$$x^2=4,$$
$$\left.\begin{aligned} x&=\pm 2 \\ y&=\pm 3 \end{aligned}\right\} Ans.$$

6. From the second equation,

$$y=40-4x,$$
$$xy=40x-4x^2;$$

hence, the first gives

$$x^2-20x=-336,$$
$$x^2-20x+400=\ 64,$$
$$x=+20\pm 8,$$
$$y=40-4x; \qquad \left.\begin{aligned} x&=\ \ 28 \text{ or } \ \ 12 \\ y&=-72 \text{ or } -\ 8 \end{aligned}\right\} Ans.$$

7. The second equation gives $x=\dfrac{5y}{2}$; whence,

$$7y^2=252,$$
$$y^2=36,$$
$$x=\tfrac{5}{2}y; \qquad \left.\begin{aligned} y&=\pm\ 6 \\ x&=\pm 15 \end{aligned}\right\} Ans.$$

8. From the second equation, $x=\frac{9}{5}y$,

$$\frac{81}{25}y^2+4y^2=181,$$
$$181y^2=181\times 25,$$
$$y^2=25,$$
$$x=\frac{9}{5}y; \qquad \left.\begin{array}{l} y=\pm 5 \\ x=\pm 9 \end{array}\right\} Ans.$$

9. Adding the equations,

$$x^2+2xy+y^2=36,$$
$$x+y=\pm 6, \qquad (1)$$

subtracting equations, $x^2-y^2=-12, \qquad (2)$

dividing (2) by (1), $x-y=\mp 2, \qquad (3)$

from (1) and (3), $\left.\begin{array}{l} x=\pm 2 \\ y=\pm 4 \end{array}\right\} Ans.$

10. Adding the square of the second equation to the first, we have

$$2x^2=5,$$
$$y=2-x; \qquad \left.\begin{array}{l} x=\pm\sqrt{\frac{5}{2}} \\ y=2\mp 2\sqrt{\frac{5}{2}} \end{array}\right\} Ans.$$

11. Put $x=vy$;

$$v^2y^2+vy^2=56, \qquad \text{or } y^2=\frac{56}{v^2+v};$$
$$vy^2+2y^2=60, \qquad \text{or } y^2=\frac{60}{v+2};$$

whence by equating the values of y^2,

$$v^2+\tfrac{1}{15}v=\tfrac{28}{15},$$
$$v=\tfrac{4}{3} \text{ or } -\tfrac{7}{5}.$$

With $v=\frac{4}{3}$, the second equation gives,

$$10y^2=180,$$
$$\left.\begin{array}{l} y=\pm\sqrt{18}=\pm 3\sqrt{2} \\ x=\phantom{\pm\sqrt{18}=}\pm 4\sqrt{2} \end{array}\right\} Ans.$$

With $v=-\frac{7}{5}$, $3y^2=300;$

$$\left.\begin{array}{l} y=\pm 10 \\ x=\mp 14 \end{array}\right\} Ans.$$

12. Let $y=vx$,

$$3x^2+vx^2=\ 68, \qquad \text{or } x^2=\frac{68}{3+v};$$

$$4v^2x^2+3vx^2=160, \qquad \text{or } x^2=\frac{160}{4v^2+3v};$$

whence, $$68v^2+11v=120,$$

or, $$4v^2+\tfrac{11}{17}v=\tfrac{120}{17},$$

$$4v^2+\frac{11}{17}v+\left(\frac{11}{68}\right)^2=\frac{32761}{(68)^2},$$

$$2v+\tfrac{11}{68}=\pm\tfrac{181}{68},$$

and, $$v=\tfrac{5}{4} \text{ or } -\tfrac{24}{17}.$$

With the value $v=\frac{5}{4}$, the first equation gives

$$17x^2=68\times 4,$$

$$x^2=16; \qquad \left.\begin{array}{l} x=\pm 4 \\ y=\pm 5 \\ x=\pm\dfrac{34}{3\sqrt{3}} \\ y=\mp\dfrac{16}{\sqrt{3}} \end{array}\right\} Ans.$$

The value of $v=-\frac{24}{17}$, gives

13. Let $y=vx$;

$$x^2+vx^2=12, \qquad \text{or } x^2=\frac{12}{1+v};$$

$$vx^2-2v^2x^2=\ 1, \qquad \text{or } x^2=\frac{1}{v-2v^2};$$

whence, $$24v^2-11v=-1,$$

$$v=\tfrac{1}{3} \text{ or } \tfrac{1}{8}.$$

Hence, if $v=\frac{1}{3}$, $\quad 4x^2=36, \qquad x=\pm 3, \quad y=\pm 1$

If $v=\frac{1}{8}$, $\quad 9x^2=96$

$$x=\pm\frac{4\sqrt{6}}{3}, \quad \text{or } x=\pm\frac{8}{\sqrt{6}}, \qquad y=\pm\frac{1}{\sqrt{6}} \quad \Bigg\} Ans.$$

14. Put $y=vx$;

$$x^2-\ vx^2+v^2x^2=\ \ 21, \qquad \text{or } x^2=\frac{21}{1-v+v^2};$$

$$v^2x^2-2vx^2=-15, \qquad \text{or } x^2=\frac{-15}{v^2-2v};$$

whence,

$$36v^2-57v=-15,$$
$$v=\ \tfrac{5}{4} \text{ or } \tfrac{1}{3}.$$

From the second equation,

$$\text{if } v=\tfrac{5}{4}, \qquad y^2-\tfrac{8}{5}y^2=-15; \qquad \left.\begin{array}{l} y=\pm 5 \\ x=\pm 4 \\ y=\pm\sqrt{3} \\ x=\pm 3\sqrt{3} \end{array}\right\} \textit{Ans.}$$
$$\text{if } v=\tfrac{1}{3}, \qquad y^2-6y^2=-15;$$

15. Add 1 to the first equation, and take the square root, then

$$x+y=+10 \text{ or } -12,$$
$$xy-y^2=\ \ 8.$$

Taking $x+y=+10$, the second equation gives, by substitution,

$$y^2-5y=-4;$$

whence,
$$\left.\begin{array}{l} y=4 \text{ or } 1 \\ x=6 \text{ or } 9 \end{array}\right\} \textit{Ans.}$$

Taking $x+y=-12$, we have

$$y^2+6y=-4;$$

whence,
$$\left.\begin{array}{l} y=-3\pm\sqrt{5} \\ x=-9\mp\sqrt{5} \end{array}\right\} \textit{Ans.}$$

16. Put $y=vx$; then

$$6x^2+2v^2x^2-5vx^2=12, \qquad \text{or } x^2=\frac{12}{6+2v^2-5v};$$

$$3v^2x^2-3x^2-2vx^2=\ 3, \qquad \text{or } x^2=\frac{3}{3v^2-3-2v}.$$

Hence,
$$10v^2-3v=18,$$
$$v=\tfrac{3}{2} \text{ or } -\tfrac{6}{5},$$

If we take $v=\frac{3}{2}$, the first equation gives

$$3x^2=12,$$

whence,
$$\left.\begin{array}{l} x=\pm 2 \\ y=\pm 3 \end{array}\right\} \textit{Ans.}$$

If $v=-\frac{6}{5}$, we have $372x^2=300$,

$$x^2=\frac{300}{372}=\frac{3\cdot4\cdot25}{3\cdot4\cdot31};$$

whence,

$$\left.\begin{aligned} x&=\pm\frac{5}{\sqrt{31}} \\ y&=\mp\frac{6}{\sqrt{31}} \end{aligned}\right\} Ans.$$

17. Multiply the second equation by 2, and add the product to the first; then

$$x^2+2xy+y^2=121,$$

subtract it; $x^2-2xy+y^2=9$,

whence, $x+y=\pm11$,

$$x-y=\pm3; \text{ whence, } \left.\begin{aligned} x&=\pm7 \text{ or } \pm4 \\ y&=\pm4 \text{ or } \pm7 \end{aligned}\right\} Ans.$$

18. Squaring the second equation, and subtracting the square from the first, we have $2xy=80$; adding this to the first, and taking the square root,

$$x+y=\pm13,$$

$$x-y=3; \text{ whence, } \left.\begin{aligned} x&=+8 \text{ or } -5 \\ y&=+5 \text{ or } -8 \end{aligned}\right\} Ans.$$

19. Dividing the first equation by the second,

$$x^2-xy+y^2=273,$$

square of second, $x^2+2xy+y^2=324$, (1)

$$3xy=51,$$

$$xy=17.$$

Subtract $4xy$ from (1),

$$x^2-2xy+y^2=256,$$

hence, $x-y=\pm16$,

$$x+y=18; \text{ whence, } \left.\begin{aligned} x&=17 \text{ or } 1 \\ y&=1 \text{ or } 17 \end{aligned}\right\} Ans.$$

20. Adding three times the second equation to the first,

$$x^3+3x^2y+3xy^2+y^3=729,$$

taking the cube root, $x+y=\ 9.$ (1)

The second equation gives, $xy(x+y)=180;$ (2)

dividing (2) by (1), $xy=\ 20.$ (2)

Subtracting four times (3) from the square of (1),

$$x^2-2xy+y^2=\ 1,$$

or, $x-y=\pm 1.$ (4)

From (1) and (4) we have $x=5$ or 4

and $y=4$ or 5 *Ans.*

21. Performing the multiplications indicated, we have

$$x^3-x^2y+\ xy^2-y^3=\ 13, \quad (1)$$

$$x^2y-\ xy^2\quad =\ \ 6; \quad (2)$$

(1) — twice (2), $x^3-3x^2y+3xy^2-y^3=\ 1,$

taking cube root, $x-y=\ 1,$ (3)

from (2) divided by (3), $xy=\ 6,$ (4)

(3) plus four times (4), $x^2+2xy+y^2=\ 25,$

$x+y=\pm 5;$ (5)

hence, from (3) and (5), $x=3$ or -2, $y=2$ or -3 *Ans.*

22. Dividing the first equation by $x+y$ and transposing, we have

$$x^2-2xy+y^2=0,$$

$$x-y=0,$$

second equation is $x+y=4;$ $x=2$, $y=2$ *Ans.*

23. The first equation is a quadratic in xy; and completing the square, we have

$$36x^2y^2-24xy+4=\ 16,$$

$$6xy-2=\pm 4,$$

$$xy=\ 1, \text{ or } -\tfrac{1}{3},$$

and since $x=2y$, $2y^2=\ 1$ or $-\frac{1}{3};$ $y=\pm\frac{1}{2}\sqrt{2}$ or $\pm\frac{1}{6}\sqrt{-6}$

$x=\pm\sqrt{2}$ or $\pm\frac{1}{3}\sqrt{-6}$

Ans.

24. Dividing the first equation by the second, we have

$$x^2-\ xy+y^2=\tfrac{3}{2}xy,$$

squaring the second, $\quad x^2+2xy+y^2=144\,; \qquad (1)$

hence, $\quad 3xy \quad =144-\tfrac{3}{2}xy,$

or, $\quad xy \quad = 32, \qquad (2)$

(1) — four times (2), $\quad x^2-2xy+y^2= 16,$

whence, $\quad x-y=\pm 4,$

$$x+y= 12\,; \qquad \left.\begin{aligned} x&=8 \text{ or } 4 \\ y&=4 \text{ or } 8 \end{aligned}\right\} Ans.$$

25. We make use of formula (C), page 242, in which

$$s=8, \quad s^2=64, \quad s^4=4096.$$

$$\begin{aligned} 2p^2-256p+4096&= \ \ 2402, \\ p^2-128p&=-\ 847, \\ p^2-128p+4096&= \ \ 3249, \\ p-64&=\pm\ \ 57, \\ xy=p&= \ \ 121 \quad \text{or } 7. \end{aligned}$$

Taking $xy=7$, and combining it with $x+y=8$, we get

$$x-y=\pm 6\,; \qquad \left.\begin{aligned} x&=7 \quad \text{or } 1 \\ y&=1 \quad \text{or } 7 \end{aligned}\right\} Ans.$$

If $xy=121, \quad x-y=\pm\sqrt{-420}=\pm 2\sqrt{-105}\,;$

whence,

$$\left.\begin{aligned} x&=4\pm\sqrt{-105} \\ y&=4\mp\sqrt{-105} \end{aligned}\right\} Ans.$$

26. Dividing the first equation by the second, we have

$$\frac{x^3+x}{x^3+1}=\frac{x(x+1)}{x^3+1}=\frac{2}{3},$$

or,

$$\frac{x}{x^2-x+1}=\frac{2}{3}\,;$$

whence, $\quad 2x^2-5x=-2,$

$$x= \ \tfrac{5}{4}\pm\tfrac{3}{4}\,; \qquad \left.\begin{aligned} x&=2 \quad \text{or } \tfrac{1}{2} \\ y&=2 \quad \text{or } 16 \end{aligned}\right\} Ans.$$

27. Dividing the first equation by $x+y$,

$$\begin{aligned} x^2-\ xy+y^2&=2xy, \\ x^2-2xy+y^2&=xy=16\,; \\ x-y&=\pm\ 4, \\ x^2+2xy+y^2&=\ 80, \\ x+y&=\pm 4\sqrt{5}\,; \end{aligned}$$

whence, $\left.\begin{aligned} x&=\pm 2\sqrt{5}\pm 2 \\ y&=\pm 2\sqrt{5}\mp 2 \end{aligned}\right\}$ *Ans.*

28. Put $x^{\frac{3}{4}}=P$, $y^{\frac{3}{4}}=Q$; then the equations become,

$$\begin{aligned} P^2+PQ&=a, \\ Q^2+PQ&=b\,; \end{aligned}$$

whence, $P+Q=\sqrt{a+b}$, (1)

and $P^2-Q^2=a-b$,

and $P-Q=\dfrac{a+b}{\sqrt{a+b}}$; (2)

by (1) and (2), $2P=\dfrac{a-b}{\sqrt{a+b}}+\sqrt{a+b}$,

$$2Q=\sqrt{a+b}+\frac{b-a}{\sqrt{a+b}}\,;$$

and we obtain finally, $P=\dfrac{a}{\sqrt{a+b}}$, or $x=\left(\dfrac{a^4}{(a+b)^2}\right)^{\frac{1}{3}}$,

$Q=\dfrac{b}{\sqrt{a+b}}$, or $y=\left(\dfrac{b^4}{(a+b)^2}\right)^{\frac{1}{3}}$.

29. Dividing the second equation by the first, we have

$$\frac{(x-y)^2}{x-y-1}=\ 4,$$

or, $(x-y)^2-4(x-y)=-4$,

$$(x-y)^2-4(x-y)+4=\ 0,$$

hence, $x-y=\ 2$.

Hence the second gives $x+y=\ 8$;

$\left.\begin{aligned} x&=5 \\ y&=3 \end{aligned}\right\}$ *Ans.*

30. Adding and subtracting twice the second equation,

$$x^2+2xy+y^2=a+2b,$$
$$x^2-2xy+y^2=a-2b;$$
$$x+y=\pm\sqrt{a+2b},$$
$$x-y=\pm\sqrt{a-2b};$$
$$\left.\begin{aligned}x&=\pm\tfrac{1}{2}\sqrt{a+2b}\pm\tfrac{1}{2}\sqrt{a-2b}\\ y&=\pm\tfrac{1}{2}\sqrt{a+2b}\mp\tfrac{1}{2}\sqrt{a-2b}\end{aligned}\right\}\ \textit{Ans.}$$

31. We have $x=\dfrac{2b}{y}$; whence,

$$\frac{4b^2}{y^2}-y^2=2a,$$
$$y^4+2ay^2=4b^2,$$
$$y^4+2ay^2+a^2=a^2+4b^2;$$

whence, $y=\pm\left\{-a\pm\sqrt{a^2+4b^2}\right\}^{\frac{1}{2}}$

in the same way, $x=\pm\left\{\ \ a\pm\sqrt{a^2+4b^2}\right\}^{\frac{1}{2}}$ *Ans.*

32. Squaring the first equation, we have

$$y^2x+2y\sqrt{xy}+y=441;$$
$$y^2x+y=333, \qquad (1)$$
$$y\sqrt{xy}=\ 54. \qquad (2)$$

Dividing (1) by (2), we have a quadratic in xy,

$$6xy-37\sqrt{xy}=-6,$$

whence, $xy=\ 36$ or $\frac{1}{36}$.

Substituting $xy=36$ in (1), $37y=333$; $y=\ 9$, $x=\ 4$

" $xy=\frac{1}{36}$ in (1), $37y=333\cdot36$; $y=324$, $x=\dfrac{1}{(108)^2}$ *Ans.*

33. Adding the equations,

$$x+2\sqrt{xy}+y=a+b,$$
$$\sqrt{x}+\sqrt{y}=\sqrt{a+b}; \qquad (1)$$

subtracting them $\qquad x-y=a-b,$

$$\sqrt{x}-\sqrt{y}=\frac{a-b}{\sqrt{a+b}}. \qquad (2)$$

Hence, from (1) and (2), $\qquad \left.\begin{aligned} x&=\frac{a^2}{a+b} \\ y&=\frac{b^2}{a+b} \end{aligned}\right\} Ans.$

34. Put $x^{\frac{1}{3}}=P, \quad y^{\frac{1}{3}}=Q,$ then

$$P^3+Q^3=35,$$
$$P+Q=5.$$

The solution is the same as that of 19.

35. Clearing of fractions, the second equation gives

$$x+y=\tfrac{5}{2}\sqrt{xy},$$

the first is, $\qquad x+y=10;$

whence, $\qquad xy=16.$

Combining the two last equations, $\qquad \left.\begin{aligned} x&=8 \text{ or } 2 \\ y&=2 \text{ or } 8 \end{aligned}\right\} Ans.$

36. Dividing the second equation by the first,

$$\frac{x-y}{4(x^{\frac{1}{2}}-y^{\frac{1}{2}})}=\frac{16}{x^{\frac{1}{2}}+y^{\frac{1}{2}}},$$

or, $$\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{4}=\frac{16}{x^{\frac{1}{2}}+y^{\frac{1}{2}}},$$

whence, $$x^{\frac{1}{2}}+y^{\frac{1}{2}}=\pm 8, \qquad (1)$$
$$x-y=16,$$

by division, $$x^{\frac{1}{2}}-y^{\frac{1}{2}}=\pm 2. \qquad (2)$$

Taking the positive signs in (1) and (2), we have

$$x^{\frac{1}{2}}=5, \qquad y^{\frac{1}{2}}=3;$$

negative signs, $\qquad x^{\frac{1}{2}}=-5, \qquad y^{\frac{1}{2}}=-3; \qquad \left.\begin{aligned} x&=25 \\ y&=9 \end{aligned}\right\} Ans.$

37. By transposition, and squaring the second equation, we have

$$y^{\frac{2}{3}}=x^2-2x^{\frac{3}{2}}+x;$$

and substituting this value of $y^{\frac{2}{3}}$ in the first equation,

$$x^2-x^{\frac{3}{2}}=2x,$$
$$x-\sqrt{x}=2,$$
$$\sqrt{x}=\tfrac{1}{2}\pm\tfrac{3}{2},$$
$$\sqrt{x}=2 \text{ or } -1;$$

From the second equation,

$$\left.\begin{aligned} x&=4 \text{ or } 1 \\ y&=8 \end{aligned}\right\} \textit{Ans.}$$

38. Adding twice the second equation to the first, we have

$$(x^{\frac{1}{3}}+y^{\frac{1}{3}})^2+2(x^{\frac{1}{3}}+y^{\frac{1}{3}})=35,$$

From this quadratic in $x^{\frac{1}{3}}+y^{\frac{1}{3}}$,

$$x^{\frac{1}{3}}+y^{\frac{1}{3}}=-1\pm 6, =5 \text{ or } -7;$$

combining this with the value of $x^{\frac{1}{3}}y^{\frac{1}{3}}$,

$$x^{\frac{1}{3}}-y^{\frac{1}{3}}=\pm 1, \text{ or } \pm 5;$$

whence,

$$\left.\begin{aligned} x&=27,\ 8,\ -1,\ -216 \\ y&=\ 8,\ 27,\ -216,\ -1 \end{aligned}\right\} \textit{Ans.}$$

39. Put $P=x^{\frac{2}{3}}$, $Q=y^{\frac{1}{5}}$, and the equations become,

$$P^2+Q^2+P+Q=26, \qquad (1)$$
$$PQ=8. \qquad (2)$$

Adding twice the second to the first,

$$(P+Q)^2+(P+Q)=42,$$
$$(P+Q)^2+(P+Q)+\tfrac{1}{4}=\tfrac{169}{4},$$
$$P+Q=6 \text{ or } -7, \qquad (3)$$
$$P^2-2PQ+Q^2=4 \text{ or } 17,$$
$$P-Q=\pm 2 \text{ or } \pm\sqrt{17}; \qquad (4)$$

hence from (3) and (4),

$$P=x^{\frac{2}{3}}=\ 4,\ 2 \text{ or } \tfrac{1}{2}\left(-7\pm\sqrt{17}\right),$$
$$Q=y^{\frac{1}{5}}=\ 2,\ 4 \text{ or } \tfrac{1}{2}\left(-7\pm\sqrt{17}\right);$$

whence,

$$\left.\begin{aligned} x&=\pm 8,\ \pm 2\sqrt{2} \text{ or } \pm\left\{\tfrac{1}{2}\left(-7\pm\sqrt{17}\right)\right\}^{\frac{3}{2}} \\ y&=\ 32,\ 1024 \text{ or } \left\{\tfrac{1}{2}\left(-7\mp\sqrt{17}\right)\right\}^5 \end{aligned}\right\} \textit{Ans.}$$

40. The first equation is a quadratic in $\frac{x}{y}$; and adding 4 to each side, and taking the square root,

$$\frac{\sqrt{x}}{\sqrt{y}}+2=\pm\frac{7}{2},$$

$$\frac{\sqrt{x}}{\sqrt{y}}=\frac{3}{2} \text{ or } -\frac{11}{2},$$

$$4x=9y \text{ or } 121y.$$

From the second equation, $\left.\begin{array}{l} x=9 \text{ or } \frac{605}{117} \\ y=4 \text{ or } \frac{20}{117} \end{array}\right\}$ *Ans.*

41. Here $P=x^{\frac{1}{3}}$, $Q=y^{\frac{1}{3}}$,

$$P^2Q^2=2Q^4, \qquad (1)$$

$$8P-Q=14; \qquad (2)$$

from (1), $$P^2=2Q, \qquad (3)$$

" (3) and (2), $$P^2-16P=-28,$$

hence, $$x^{\frac{1}{3}}=P=8\pm6=14 \text{ or } 2,$$

$$y^{\frac{1}{3}}=Q= \quad 98 \text{ or } 2;$$

whence, $\left.\begin{array}{l} x=2744 \text{ or } 8 \\ y=9604 \text{ or } 4 \end{array}\right\}$ *Ans.*

42. Let $P=x^{\frac{3}{4}}$, $Q=y^{\frac{3}{4}}$; then

$$P^2+PQ+Q^2= \quad 1009=a, \qquad (1)$$

$$P^4+P^2Q^2+Q^4=582193=577a,$$

Add P^2Q^2, and take the root, $$P^2+Q^2=\sqrt{577a+P^2Q^2} \qquad (2)$$

subtract (1) from (2), $$a-PQ=\sqrt{577a+P^2Q^2},$$

$$a^2-2aPQ+P^2Q^2=577a+P^2Q^2,$$

$$2PQ=a-577,$$

$$PQ=216, \qquad (3)$$

Adding (3) to (1) and taking the root, $$P+Q=\pm35,$$

subtracting three times (3) from (1), $$P-Q=\pm19;$$

(246)

hence, $x^{\frac{3}{4}} = P = \quad 8 \text{ or } -27,$

$$y^{\frac{3}{4}} = Q = -27 \text{ or } \quad 8,$$

whence, $\left.\begin{array}{l} x = 81 \text{ or } 16 \\ y = 16 \text{ or } 81 \end{array}\right\} Ans.$

43. Complete the square of each equation by adding $16x$ to the first, and x to the second; then

$$y^2 - 8x^{\frac{1}{2}}y + 16x = 16(4+x),$$

$$y - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + \ x = \quad 4+x;$$

taking the roots, $y - 4\sqrt{x} = \pm 4\sqrt{4+x}, \qquad (1)$

$$\sqrt{y} - \sqrt{x} = \pm\sqrt{4+x}; \qquad (2)$$

(1) minus four times (2), $y - 4\sqrt{y} = \quad 0;$

whence, $\left.\begin{array}{l} y = 16 \\ x = \ 2\frac{1}{4} \end{array}\right\} Ans.$

44. These equations may be written,

$$xy(x+y) = \ 30, \qquad (1)$$

$$\frac{x+y}{xy} = \frac{5}{6}. \qquad (2)$$

Dividing (1) by (2), $x^2y^2 = \ 36;$

$$xy = \pm 6, \qquad (3)$$

from (2) and (3), $x+y = \pm 5,$

$$x - y = \pm 1 \ \text{ or } \pm 7;$$

whence, $\left.\begin{array}{l} x = 3,\ 2,\ 1, \text{ or } -6 \\ y = 2,\ 3,\ -6, \text{ or } 1 \end{array}\right\} Ans.$

45. Dividing the first equation by the second, we have

$$x^2y^2 = \ 16,$$

$$xy = \pm 4.$$

Hence, $x^2 + 2xy + x^2 = \ 16 \ \text{ or } \quad 0,$

$$x^2 - 2xy + y^2 = \quad 0 \ \text{ or } \ 16;$$

$$x + y = \pm 4 \ \text{ or } \quad 0,$$

$$x - y = \quad 0 \ \text{ or } \pm 4;$$

whence, $\left.\begin{array}{l} x = \pm 2 \\ y = \pm 2 \end{array}\right\} \text{ or } \left\{\begin{array}{l} x = \pm 2 \\ y = \mp 2 \end{array}\right\} Ans.$

46. Dividing the first equation by the second,

$$x^4+x^3y+x^2y^2+xy^3+y^4=1031, \qquad (1)$$

$$(x-y^4)=x^4-4x^3y+6x^2y^2-4xy^3+y^4=\ 81; \qquad (2)$$

(1) minus (2), $\qquad 5x^3y-5x^2y^2+5xy^3=\ 950,$

dividing by $5xy$, $\qquad x^2-xy+y^2=\dfrac{190}{xy},$

$$(x-y)^2=x^2-2xy+y^2=\ 9, \qquad (3)$$

$$x^2y^2+9xy=\ 190,$$

$$xy=\ 10 \text{ or } -19; \qquad (4)$$

from (3) and (4), $\qquad x+y=\pm\ 7,$

whence, $\qquad \left.\begin{aligned} x&=5 \quad \text{or } -2 \\ y&=2 \quad \text{or } -5 \end{aligned}\right\}$ *Ans.*

47. We have given

$$x^2+xy+y^2=\ 7, \qquad (1)$$

$$x^4+x^2y^2+y^4=\ 133. \qquad (2)$$

Adding x^2y^2 to (2), $\qquad x^4+2x^2y^2+y^4=\ 133+x^2y^2,$

or, $\qquad x^2+y^2=\sqrt{133+x^2y^2}. \qquad (3)$

Taking (1) from (3), $\qquad 7-xy=\sqrt{133+x^2y^2},$

$$49-14xy+x^2y^2=\ 133+x^2y^2.$$

$$xy=\ -6. \qquad (4)$$

Combining (1) and (4), $\qquad x+y=\ \pm 1,$

also $\qquad x-y=\ \pm 5.$

The values of $x+y$ and $x-y$, may be combined in four ways, as follows:

$$\begin{cases} x+y=+1; & x+y=-1; & x+y=+1; & x+y=-1; \\ x-y=+5; & x-y=\ 5; & x-y=-5; & x-y=-5; \end{cases}$$

Giving $\begin{cases} x=+3; & x=+2; & x=-2; & x=-3; \\ y=-2; & y=-3; & y=+3; & y=+2. \end{cases}$

Or, condensing the expression,

$$\left.\begin{aligned} x&=\pm 2 \quad \text{or } \mp 3 \\ y&=\mp 3 \quad \text{or } \pm 2 \end{aligned}\right\} \textit{Ans.}$$

48. Eliminating y from the last two equations, we have

$$52x^2-36x=-5,$$
$$x^2-\tfrac{9}{13}x+\tfrac{81}{676}=\tfrac{16}{676},$$
$$x=\tfrac{9}{26}\pm\tfrac{4}{26};$$

whence,

$$y=\frac{5-8x}{3}; \qquad \left.\begin{array}{l} x=\tfrac{1}{2} \text{ or } \tfrac{5}{26} \\ y=\tfrac{1}{3} \text{ or } \tfrac{15}{13} \\ z=\tfrac{1}{4} \text{ or } \tfrac{15}{44} \end{array}\right\} \textit{Ans.}$$

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

(314, page 258.)

1. Let $x=$ the greater part; then $14-x=$ the less. Hence,

$$\frac{9x}{14-x}=\frac{16(14-x)}{x},$$
$$9x^2=16(14-x)^2,$$
$$3x=4(14-x),$$
$$x=8; \qquad\qquad 8 \text{ and } 6, \textit{ Ans.}$$

2. Let $x=$ the number of persons; then

$$\frac{350}{x}+20=\frac{350}{x-2}.$$

Hence,

$$x^2-2x=35,$$
$$x^2-2x+1=36,$$
$$x-1=6; \qquad\qquad x=7, \textit{ Ans.}$$

3. Let $x=$ the number; then

$$(22-x)x=117,$$

or,

$$x^2-22x=-117,$$
$$x^2-22x+121=4,$$
$$x=11\pm2; \qquad\qquad x=13 \text{ or } 9, \textit{ Ans.}$$

4. Let $x=$ one part, and $18-x=$ the other; then

$$\frac{x^2}{(18-x)^2}=\frac{25}{16},$$

$$\frac{x}{18-x}=\frac{5}{4},$$

$$9x=90; \qquad 10 \text{ and } 8, \textit{Ans.}$$

5. Let $x=$ one number, and $x-4=$ the other; then

$$(2x-4)(x-2)8=1600,$$

$$x^2-4x=\ \ 96,$$

$$x=2\pm10; \qquad 12 \text{ and } 8, \textit{Ans.}$$

6. Let $x=$ one number, and $y=$ the other, the less; then

$$x-y : y :: 4 : 3$$

or,

$$3x=7y, \qquad (1)$$

$$xy^2=504, \qquad (2)$$

$$y^3=216; \text{ whence } \quad \left.\begin{array}{l} y=\ 6 \\ x=14 \end{array}\right\} \textit{Ans.}$$

7. Take $8x$ and $5x$ respectively, for the length and breadth of the field; then

$$\frac{8x\times5x}{160}= \text{the number of acres } =\frac{x^2}{4}.$$

Hence,

$$\frac{x^2}{4}\times8=13\times26x,$$

$$x^2=13\times13,$$

$$x=13; \qquad \left.\begin{array}{l} 8x=104, \text{ length} \\ 5x=\ \ 65, \text{ breadth} \end{array}\right\} \textit{Ans.}$$

8. Take $5x$ and $4x$ for the length and breadth of the stack; $\frac{7}{2}x$ will be the height. Hence there are $70x^3$ cubic feet in the stack, and $20x^2$ square feet on the bottom. Therefore,

$$70x^3\times4x=224\times20x^2,$$

$$x^2=16,$$

$$x=4,$$

Length $=5x=20$; Breadth $=4x=16$; Height $=\frac{7}{2}x=14$, *Ans.*

9. Let x^2-7 be the number; then from the statement of the problem, we have

$$x+\sqrt{x^2+9}=9,$$

or,

$$x^2+9=81-18x+x^2,$$

$$x=4; \qquad x^2-7=9, \text{ Ans.}$$

10. Let x=A's number of eggs; then $\frac{18}{100-x}$=A's price per egg;

$100-x$=B's " " ; $\frac{8}{x}$ =B's " "

Hence, since they received the same amount of money, if we multiply each man's number of eggs by his price per egg, we have the equation,

$$\frac{18x}{100-x}=\frac{8(100-x)}{x};$$

or,

$$9x^2=4(100-x)^2,$$

$$3x=2(100-x),$$

$$x=40; \qquad \text{A, 40; B, 60, Ans.}$$

11. Here we have from the statement,

$$x+y=6,$$

$$x^2+y^2=72,$$

and the solution is like that of 19, (**301**).

12. Let x= the number of miles per hour; then $\frac{36}{x}$= the number of hours. Whence from the conditions given, we have

$$\frac{36}{x}=\frac{36}{x+1}+3;$$

or,

$$x^2+x=12,$$

$$x^2+x+\tfrac{1}{4}=\tfrac{49}{4}; \qquad x=3, \text{ Ans.}$$

13. Let x and y be the numbers; we have from the statement,

$$x+y=100, \qquad (1)$$

$$\sqrt{x}-\sqrt{y}=2;$$

squaring, $$x-2\sqrt{xy}+y=4, \qquad (2)$$

from (1) and (2), $$\sqrt{xy}=48, \qquad (3)$$

four times (3)+(2), $$x+2\sqrt{xy}+y=196,$$

$$\sqrt{x}+\sqrt{y}=14,$$

$$\sqrt{x}=8; \qquad x=64, y=36, \text{ Ans.}$$

14. Let $x=$ the number of pieces; then $\frac{675}{x}=$ cost of one piece, and from the conditions given,

$$48x=675+\frac{675}{x},$$

$$48x^2-675x=675,$$

or, $$16x^2-225x=225; \qquad x=15, \textit{Ans.}$$

See solution of Example 3, Art. (**295**),

15. Let $x=$ the price of cloth; then $\frac{x}{100}=$ gain per cent., and $\frac{x^2}{100}=$ the whole gain. Whence,

$$x+\frac{x^2}{100}=39,$$

$$x^2+100x=3900,$$

$$x=-50\pm80; \qquad x=\$30, \textit{Ans.}$$

16. Let $x=$ the purchase money; then $x+\frac{4x}{100}=\frac{104x}{100}=$the whole cost, and $390-\frac{104x}{100}=$ the gain. By the statement we have also the gain $=\frac{104x}{100}\cdot\frac{x}{12}\cdot\frac{1}{100}$. Hence,

$$390-\frac{104x}{100}=\frac{104x^2}{120000},$$

or,

$$3000-8x=\frac{2x^2}{300}.$$

Put $a=300$, and divide by 2;

$$5a-4x=\frac{x^2}{a},$$

$$x^2+4ax+4a^2=9a^2,$$

$$x=a; \qquad x=\$300, \textit{Ans.}$$

17. Observe that $396-216=180$ miles, B's distance. Let $x=$ the number of days they traveled; then $\frac{216}{x}=$ A's rate, and $\frac{180}{x}=$ B's rate. Hence,

$$x=\frac{216}{x}-\frac{180}{x},$$
$$x^2=36,$$
$$x=6; \qquad \text{A, 36; B, 30, } \textit{Ans.}$$

18. With two unknown quantities, $x+y=60$, $xy=704$; See (**299**), Example 3. With one unknown quantity,

$$60x-x^2=704,$$
$$x^2-60x+900=196; \qquad x=44 \text{ or } 16, \textit{ Ans.}$$

19. Let $x=$ price of sherry per dozen,
$y=$ " claret "

Then $$7x+12y=50;$$
$$\frac{10}{x}=\frac{6}{y}+3,$$

or, $$x=\frac{10y}{3y+6}.$$

By substitution, $$70y+36y^2+72y=150y+300,$$
$$9y^2-2y=75; \qquad y=3,\ x=2, \textit{ Ans.}$$

20. Let $19x=$ the distance from C to D;
then, $x=$ B's rate per day, and also his number of days,
$x^2=$ B's distance,
$7x+32=$ A's distance.

Hence we have,

$$x^2+7x+32=19x,$$
$$x^2-12x=-32,$$
$$x=6\pm2,$$
$$x=8 \text{ or } 4; \qquad 19x=152 \text{ or } 76, \textit{ Ans.}$$

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21. Let $x=$ the bushels of wheat,

$x+16=$ " " barley.

Hence, $$\frac{24}{x}=\frac{24}{x+16}+\frac{1}{4}, \quad \text{(25 cents} =\tfrac{1}{4} \text{ of a dollar.)}$$

$$24x+16\cdot24=24x+\frac{x^2+16x}{4},$$

$$\frac{x^2}{4}+4x=384,$$

$$\frac{x^2}{4}+4x+16=400,$$

$$\frac{x}{2}=-4\pm20; \qquad 32, \text{ and } 48, \textit{Ans.}$$

22. Let $2x=$ the distance from C to D,

$x+18=$ A's distance, $\quad \frac{4(x-18)}{63}=$ A's rate per day;

$x-18=$ B's " $\quad \frac{x+18}{28}=$ B's " "

The distance A traveled divided by his rate per day gives the number of days he traveled; and since B traveled the same number of days, we have the equation,

$$\frac{63(x+18)}{4(x-18)}=\frac{28(x-18)}{x+18},$$

or,

$$9(x+18)^2=16(x-18)^2,$$

$$3(x+18)=4(x-18),$$

$$x=126; \qquad 2x=252, \textit{Ans.}$$

23. Let $x=$ one number; $y=$ the other; then,

$$(x-y)(x^2-y^2)=32,$$

$$(x+y)(x^2+y^2)=272;$$

or multiplying,

$$x^3-xy^2-x^2y+y^3=32,$$

$$x^3+xy^2+x^2y+y^3=272.$$

Adding and subtracting these equations and dividing by 2,

$$x^3+y^3=152, \qquad (1)$$
$$xy^2+x^2y=120, \qquad (2)$$

(1)+3·(2), $x^3+3x^2y+3xy^2+y^3=512,$

taking cube root, $x+y=\ 8.$

Dividing the first equation by this, and taking the square root,

$$x-y=\pm 2; \qquad x=5, \quad y=3, \textit{ Ans.}$$

24. Let $x=$ the number of horses B put in,

$\frac{18}{x}=$ price of one horse a week,

$\frac{4\cdot 18}{x}+18=$ rent of pasture;

when x becomes $x+2$,

$\frac{20}{x+2}=$ price of one horse a week.

$\frac{4\cdot 20}{x+2}+20=$ rent of pasture.

Therefore,

$$\frac{72}{x}+18=\frac{80}{x+2}+20,$$
$$\frac{36}{x}=\frac{40}{x+2}+\ 1,$$
$$x^2+6x=72,$$
$$x=\ 5. \qquad \text{Rent }=30 \text{ shillings, } \textit{Ans.}$$

25. Let $x=$ the figure of the ten's place,
$y=$ " " unit's " ,

$$\frac{10x+y}{xy}=\ 2, \qquad (1)$$
$$10x+y+27=\ 10y+x; \qquad (2)$$
$$9x-9y=-27,$$
$$x-y=-\ 3,$$
$$x=\ y-3.$$

Substituting in the first equation, we have

$$2y^2-17y=-30,$$
$$y=6,$$
$$x=3; \qquad 36, \textit{Ans.}$$

26. Let $x=$ the first number, which is the least; $y=$ the second, and $z=$ the third. Then

$$x-y=y-z-6,$$

or,
$$x-2y+z=-6; \qquad (1)$$
$$x+y+z=33; \qquad (2)$$
$$x^2+y^2+z^2=441. \qquad (3)$$

From (1) and (2),
$$3y=39,$$
$$y=13;$$

substituting in (1),
$$x+z=20,$$

" (3),
$$x^2+z^2=272;$$
$$x^2+2xz+z^2=400,$$
$$2xz=128,$$
$$x-z=\pm 12,$$
$$x=4,$$
$$z=16; \qquad 4, 13 \text{ and } 16, \textit{Ans.}$$

27. Let x and y be the numbers; then

$$xy=24, \qquad (1)$$
$$x+y+x^2+y^2=62. \qquad (2)$$

Adding twice the first to the second,

$$(x+y)^2+(x+y)=110,$$
$$x+y=10,$$
$$x-y=2; \qquad x=6,\ y=4, \textit{Ans.}$$

28. Let $x+y=$ one number, and $x-y=$ the other; then

$$x^2-y^2+2x=47, \qquad (1)$$
$$2x^2+2y^2-2x=62, \qquad (2)$$
$$4x^2+2x=156,$$
$$2x+\tfrac{1}{2}=\pm\tfrac{25}{2},$$
$$x=6,$$
$$y=1; \qquad 7 \text{ and } 4, \textit{Ans.}$$

(261—262)

29. Let $x=$ one number, and $y=$ the other; then

$$x+y=27,$$
$$x^3+y^3=5103.$$

Solution the same as that of Example 19, page 244.

30. If $x=$ one number, and $y=$ the other; then

$$x+y=9, \qquad (1)$$
$$x^4+y^4=2417. \qquad (2)$$

See Example 1, (**301**).

31. Let $x+y=$ one number, and $x-y=$ the other; then

$$2(x^2-y^2)(x^2+y^2)=1248, \qquad (1)$$
$$4xy=20. \qquad (2)$$

From (1), $x^4-y^4=624,$

eliminating y, $x^8-624x^4=625,$

$$x^4=625,$$
$$x=5,$$
$$y=1; \qquad 6 \text{ and } 4, \textit{Ans.}$$

32. Let $x=$ the number of days it takes one man,
$x+10=$ " " " the other; then

$$\frac{12}{x}+\frac{12}{x+10}=1,$$
$$x^2-14x=120,$$
$$x=7\pm13; \qquad 20 \text{ and } 30, \textit{Ans.}$$

33. Let

$$x=\text{A's stock},$$
$$1000-x=\text{B's} \ \text{"}$$
$$1140-x=\text{A's gain},$$
$$640-(1000-x)=x-360=\text{B's} \ \text{"}$$

Now A's stock must be to B's stock, as A's monthly gain to B's monthly gain. That is,

$$x : 1000-x=\frac{1140-x}{9} : \frac{x-360}{6}.$$

Or, by taking the products of extremes and means,

$$\frac{(x-360)x}{6}=\frac{(1140-x)(1000-x)}{9};$$

whence, $3(x^2-360x)=2(1140000-2140x+x^2)$,

$$x^2+3200x=2280000,$$
$$x+1600=2200,$$
$$x=600;$$

A's, \$600; B's, \$400, *Ans.*

34. Let $2x^2=$ the number in the first drove,

$4+4x=$ " second "

$6x^2+12x+12=$ " third "

$3x^2+\ 6x+16=$ " fourth "

Hence, we have

$$11x^2+22x+32=1121,$$

or, $x^2+2x=99$,

$$x=9,$$
$$2x^2=162;$$

therefore, 1st, 162; 2d, 40; 3d, 606; 4th, 313, *Ans.*

35. Let $x=$ one number, and $y=$ the other; then

$$3xy-x^2-y^2=11, \qquad (1)$$
$$2xy-x^2+y^2=14. \qquad (2)$$

Assume $vx=y$; then

$$3vx^2-x^2-v^2x^2=11, \qquad x^2=\frac{11}{3v-1-v^2};$$

$$2vx^2-x^2+v^2x^2=14, \qquad x^2=\frac{14}{2v-1+v^2}.$$

By equating values of x^2, we obtain

$$25v^2-20v=-3,$$
$$v=\ \tfrac{3}{5} \text{ or } \tfrac{1}{5}.$$

If $v=\frac{3}{5}$, $x^2=25$;

whence, 5 and 3, *Ans.*

36. If $x=$ one number, and $y=$ the other; then

$$x+y=\ 20, \qquad (1)$$
$$x^2y^2=9216, \qquad (2)$$
$$xy=\ 96,$$
$$x-y=\ 4; \text{ whence,} \qquad 12 \text{ and } 8, \textit{Ans.}$$

37. Let x and y be the parts; then

$$x+y=a, \qquad (1)$$
$$x^2y^2=b, \qquad (2)$$
$$xy=\sqrt{b},$$
$$x-y=(a^2-4\sqrt{b})^{\frac{1}{2}};$$

whence,

$$\left.\begin{array}{l}\frac{a}{2}+\frac{1}{2}(a^2-4\sqrt{b})^{\frac{1}{2}} \\ \frac{a}{2}-\frac{1}{2}(a^2-4\sqrt{b})^{\frac{1}{2}}\end{array}\right\} Ans.$$

This is a general solution of Example 36.

38. Let x and y be the numbers; then

$$x=a^2y, \qquad (1)$$
$$xy=b^2. \qquad (2)$$

By division,

$$y^2=\frac{b^2}{a^2}; \quad \text{or } y=\frac{b}{a},\ x=ab,\ Ans.$$

39. Let x be the first of the consecutive numbers; the number sought will be $x(x+1)(x+2)$. Hence,

$$(x+1)(x+2)+x(x+2)+x(x+1)=74$$

or,

$$x^2+2x=24,$$
$$x=4 \text{ or } -6;$$

whence,

$$\left.\begin{array}{r}4\cdot5\cdot6=\ \ 120 \\ (-6)\cdot(-5)\cdot(-4)=-120\end{array}\right\} Ans.$$

40. Let $x=$ width of the engraving; and $2x=$ the length; then $2x^2=$ square contents of the engraving,

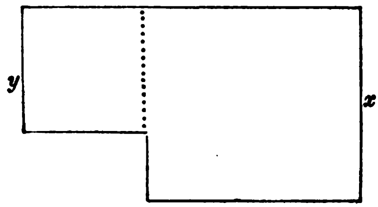
$18x+4\times3^2=$ " " margin. Hence,

$$2x^2-36=18x+36,$$
$$x^2-9x=36,$$

whence, $x=12$, *Ans.*

41. In order that the two lots may be embraced in a single inclosure of six sides, they must be placed as in the following diagram.

Let $x=$ a side of the greater square, and y a side of the smaller. Then x^2+y^2 will be the area of the two lots, and $3(x+y)+(x-y)$ or $4x+2y$, will be the length of the fence required to inclose them. Hence



$$x^2+y^2=4100, \qquad (1)$$
$$4x+2y=280; \qquad (2)$$

from (2), $\quad y=140-2x,$

substituting in (1), $\quad x^2+(140-2x)^2=4100,$

$$x^2-112x=-3100,$$
$$x-56=\pm 6,$$

$$\left.\begin{array}{l} x=62 \text{ or } 50 \\ y=16 \text{ or } 20 \end{array}\right\} \textit{Ans.}$$

42. Let $x=$ the first portion,
$a-x=$ the second portion, putting $a=1300$.
$y=$ the first rate of interest,
$z=$ the second " .

Then

$$xy=(a-x)z, \qquad (1)$$
$$xz=36, \qquad (2)$$
$$(a-x)y=49. \qquad (3)$$

We have

$$y=\frac{49}{a-x}, \qquad z=\frac{36}{x};$$

and eliminating y and z from the first equation,

$$49x^2=36(a-x)^2,$$
$$x=600;$$

whence, $\quad y=7$ per cent., $z=6$ per cent., *Ans.*

43. Let $x=$ A's number of hours,
$y=$ B's " "
$m=$ the distance from London to York;

then

$$\frac{m}{x}=\text{A's miles per hour}; \quad \frac{m}{y}=\text{B's miles per hour};$$

$$\frac{25m}{x}+\frac{36m}{y}=m,$$

or,
$$\frac{25}{x}+\frac{36}{y}=1, \qquad (1)$$

and
$$x-25=y-36, \qquad (2)$$

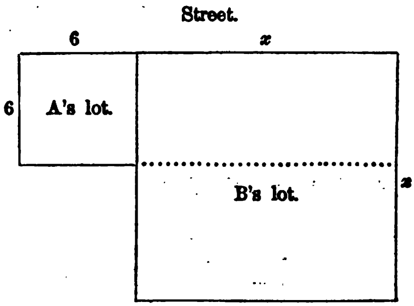
since they traveled the same number of hours before they met.

Hence, from (2), $x=y-11$,

and from (1), $y^2-72y=-396$;

therefore, $y=66, \quad x=55$, *Ans.*

44. Let $x=$ the side of B's lot.



From the conditions we have

$$36+6x=(x-6)x,$$

$$x^2+12x=36,$$

$$x=6+6\sqrt{2};$$

whence, $6(1+\sqrt{2})$, *Ans.*

45. Let x, y and z be the three numerical quantities. Then

$$x^2+y^2+x+y=32, \qquad (1)$$

$$x^2+z^2+x+z=42, \qquad (2)$$

$$y^2+z^2+y+z=50. \qquad (3)$$

Hence,
$$z^2-y^2+z-y=10,$$

$$2z^2+2z=60,$$

$$z^2+z=30,$$

$$\left.\begin{array}{l} z=5 \text{ or } -6 \\ y=4 \text{ or } -5 \\ x=3 \text{ or } -4 \end{array}\right\} \textit{Ans.}$$

46. Let $x=$ the side of the cube; then

$x^3=$ the number of solid units,

$\sqrt{3x^2}=$ the diagonal.

Therefore, $$x^3=\sqrt{3x^2},$$
$$x^6=3x^2;$$

whence, $$x=\sqrt[4]{3}, \text{ Ans.}$$

47. Let x and y be the numbers;

$$x+y=xy; \qquad (1)$$
$$x^2+y^2=xy; \qquad (2)$$
$$x^2+2xy+y^2=x^2y^2; \qquad (3)$$

from (2) and (3), $$xy=3,$$

whence, $$x-y=\pm\sqrt{-3};$$

and, $$x=\tfrac{1}{2}\left(3\pm\sqrt{-3}\right), \quad y=\tfrac{1}{2}\left(3\mp\sqrt{-3}\right), \text{ Ans.}$$

48. Let x and y be the numbers;

$$x+y=xy, \qquad (1)$$
$$x^2-y^2=xy; \qquad (2)$$

whence, $$x-y=1. \qquad (3)$$

Subtracting the square of (3) from the square of (1),

$$x^2y^2-4xy=1;$$
$$x+y=xy=2\pm\sqrt{5},$$
$$x-y=1;$$

whence, $$x=\tfrac{1}{2}(3\pm\sqrt{5}), \text{ and } y=\tfrac{1}{2}(1\pm\sqrt{5}), \text{ Ans.}$$

49. Let x and y represent the numbers; then

$$x^3-y^3=xy, \qquad (1)$$
$$x^3-y^3=x^2+y^2. \qquad (2)$$

Put $x=vy$; then the equations become

$$v^3y^3-y^3=vy^2, \qquad (3)$$
$$v^3y^3-y^3=v^2y^2+y^2. \qquad (4)$$

Dividing each equation by y^2,

$$v^2-1=v, \qquad (5)$$
$$(v^3-1)y=v^2+1. \qquad (6)$$

From (5) we have, by the rule for quadratics,

$$2v=1\pm\sqrt{5}. \qquad (7)$$

Multiplying (5) by v, $\quad v^3-v=v^2\,;$ (8)

transposing in (5), $\quad v^2=v+1\,;$ (9)

whence, by (8) and (9), $\quad v^3-v=v+1,$

or, $\quad v^3-1=2v.$ (10)

Putting this value of (v^3-1) in (6), we have

$$2vy=v^2+1=v+2,$$

or, $\quad 4vy=2v+4\,;$

from (7), $\quad 2(1\pm\sqrt{5})y=5\pm\sqrt{5}\,;$

whence, $\quad y=\pm\frac{1}{2}\sqrt{5}$

and $\quad x=vy=\frac{1}{2}(1\pm\sqrt{5})\cdot\frac{1}{2}\sqrt{5}\,;$ or $\quad x=\frac{1}{4}(5\pm\sqrt{5})$

$$\left.\begin{array}{l} y=\pm\frac{1}{2}\sqrt{5} \\ x=\frac{1}{4}(5\pm\sqrt{5}) \end{array}\right\} \textit{Ans.}$$

ANOTHER METHOD.

$$x^2-y^2=xy, \qquad (1)$$
$$x^3-y^3=x^2+y^2. \qquad (2)$$

From (1), $\quad x^2-xy=y^2\,;$ (3)

completing square, $\quad x^2-xy+\dfrac{y^2}{4}=\dfrac{5y^2}{4},$

or, $\quad x-\frac{1}{2}y=\pm\frac{1}{2}\sqrt{5}\cdot y,$

or, $\quad x=\frac{1}{2}(1\pm\sqrt{5})y\,;$

by involution, $\quad x^2=\frac{1}{2}(3\pm\sqrt{5})y^2,$

$$x^3=(2\pm\sqrt{5})y^3.$$

Substituting the values of x^2 and x^3 in (2), we have

$$(2\pm\sqrt{5})y^3-y^3=\tfrac{1}{2}(3\pm\sqrt{5})y^2+y^2,$$
$$(2\pm\sqrt{5})y-y=\tfrac{1}{2}(3\pm\sqrt{5})+1,$$
$$(1\pm\sqrt{5})y=\tfrac{1}{2}(5\pm\sqrt{5})\,;$$

$$\left.\begin{array}{l} y=\pm\frac{1}{2}\sqrt{5} \\ x=\frac{1}{4}(5\pm\sqrt{5}) \end{array}\right\} \textit{Ans.}$$

PROBLEMS IN PROPORTION.

(**334**, page 276.)

6. Let x and y be the numbers.

$$x+4 : y+4 = 3 : 4,$$
$$x-4 : y-4 = 1 : 4.$$

Therefore,

$$3y-4x = 4,$$
$$y-4x = -12,$$

whence, $2y = 16;\qquad y=8,\quad x=5,$ *Ans.*

7. Let x and y be the numbers.

$$x+y=27,$$
$$xy : x^2+y^2=20 : 41.$$

Multiply the antecedents by 2, and we have, by composition and division,

$$(x+y)^2 : 2xy = 81 : 40,$$
$$(x-y)^2 : 2xy = 1 : 40;$$

whence, $(x+y)^2 : (x-y)^2 = 81 : 1,$

or, $x+y : x-y = 9 : 1,$

or, $27 : x-y = 9 : 1,$

or, $x-y = 3;$

$$x+y = 27;$$

whence, $x=15,\quad y=12,$ *Ans.*

8. Let $x=$ the number of gallons of rum,
$y=$ " " " brandy.

$$x-y : y = 100 : x,$$
$$x-y : x = 4 : y.$$

By Proposition XI,

$$(x-y)^2 : xy = 400 : xy,$$

or, $(x-y)^2 : 1 = 400 : 1;$

therefore, $x-y = 20.$

Again, $1 : \frac{y}{x} = 25 : \frac{x}{y}$,

or, $1 : 25 = y^2 : x^2$,

$1 : 5 = y : x$,

therefore, $5y = x$;

whence, $y=5, \quad x=25$, *Ans.*

9. Let $x+y=$ the greater,

$x-y=$ the less.

$$(x+y)^3=x^3+3x^2y+3xy^2+y^3,$$
$$(x-y)^3=x^3-3x^2y+3xy^2-y^3.$$

Hence,

$$6x^2y+2y^3 : 8y^3 = 61 : 1,$$
$$3x^2+y^2 : 4y^2 = 61 : 1;$$

therefore, $3x^2+y^2=244y^2$,

$$x^2 = 81y^2.$$

But, $x^2-y^2=320$,

$$80y^2=320,$$
$$y = 2,$$
$$x = 18;$$

whence, 20 and 16, *Ans.*

NOTE.—See Example 4 for a different solution.

10. Let x and y be the numbers.

$$xy : x^2+y^2 = 2 : 5,$$

or, $2xy : x^2+y^2 = 4 : 5$;

by Prop. VI, $(x+y)^2 : (x-y)^2 = 9 : 1$. (1)

We have given, $x+y=60$;

whence, by (1), $x-y=20$;

therefore, $x=40, \quad y=20$, *Ans.*

11. Let $3x$ and $2x$ be the numbers.

$$3x+6 : 2x-6 = 3 : 1,$$

whence, $x=8$; 24 and 16, *Ans.*

12. Take $16x$ and $9x$ for the numbers.

$$16x : 24 = 24 : 9x,$$

$x = 2$; therefore, 32 and 18, *Ans.*

(276—277)

13. Let x and y be the numbers.

$$x+y : x-y = 4 : 1;$$

by Prop. VI, $$x : y = 5 : 3;$$

$$5y=3x.$$

Again, $$x^2+y^2 : x = 102 : 5,$$

$$x^2+\tfrac{9}{25}x^2 : x = 102 : 5,$$

or, $$34x : 25 = 102 : 5;$$

whence, $x=15$, $y=9$, *Ans.*

14. Let x and y be the parts.

$$x : y = 9 : 1,$$

$$9y = x;$$

$$x+y=20;$$

whence, $$y=2,$$

$x=18$. Hence, $\sqrt{xy}=6$, *Ans.*

15. Let $3x$ and $2x$ be the numbers.

$$3x+6 : 2x-6 = 9 : 4,$$

$$x=13;$$

whence, 39 and 26, *Ans.*

16. Take x, xy and xy^2 for the numbers.

$$x^2y : x^2y^2 = x : 2xy^2,$$

or, $$1 : y^2 = 1 : 2y;$$

whence, $$y=2.$$

By the conditions, $$x+xy^2=300,$$

whence, $5x=300$; therefore, 60, 120 and 240, *Ans.*

17. Let x and y be the two numbers,

$$x^3+y^3 : x^3-y^3 = 559 : 127;$$

Prop. VI, $$2x^3 : 2y^3 = 686 : 432,$$

or, $$x^3 : y^3 = 343 : 216,$$

or, $$x : y = 7 : 6;$$

whence, $$6x = 7y.$$

By the conditions, $$x^2y=294;$$

$x^3=7\cdot 49$; whence, 7 and 6, *Ans.*

18. Let x and y be the numbers.

$$x^3 : y^2 = 3 : 1, \qquad (1)$$
$$y^3 : x^2 = 96 : 1; \qquad (2)$$
$$3y^2 = x^3, \qquad (3)$$
$$y^3 = 96x^2; \qquad (4)$$

from (3), $\quad y=\dfrac{x^{\frac{3}{2}}}{3^{\frac{1}{2}}},$

whence from (4), $\quad x^{\frac{9}{2}}=3^{\frac{3}{2}}\cdot 96x^2,$

or, $\quad x^{\frac{5}{2}}=3^{\frac{3}{2}}\cdot 96,$

or, $\quad x^5=3^5\cdot(32)^2;$

therefore, $\quad x=12$, and $y=24$, *Ans.*

19. By Prop. X, this proportion becomes

$$(x+1)^2 : (x-1)^2 = 2 : 1;$$

whence, $\quad x\sqrt{2}-\sqrt{2}=x+1,$

and, $\quad x=\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$, *Ans.*

20. Putting the given equation in the form of a proportion, we have

$$a+b+c+d : a-b+c-d = a+b-c-d : a-b-c+d;$$

by Prop. VI, and dividing by 2,

$$a+c : b+d = a-c : b-d,$$

or, $\quad a+c : a-c = b+d : b-d,$

Again by Prop. VI, $\quad a : c = b : d,$

or, $\quad a : b = c : d$, *Ans.*

PERMUTATIONS AND COMBINATIONS.

(**341**, page 283.)

1. Here $\quad n-r+1=10-4+1=7$; hence,

$$10\cdot 9\cdot 8\cdot 7=5040, \text{ Ans.}$$

2. $\quad 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1=720$, *Ans.*

3. By formula (B),

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800, \textit{ Ans.}$$

4. Omitting the 0, the other four figures can be arranged in

$$4 \cdot 3 \cdot 2 \cdot 1 = 24 \text{ ways.}$$

Now we must reject every combination formed by placing the cipher *before* all the other figures. Hence, in each of the 24 combinations of the figures 4, 3, 2, 1, the cipher may have 4 places. Therefore,

$$24 \times 4 = 96, \textit{ Ans.}$$

5. By formula (C),

$$\frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70, \textit{ Ans.}$$

6. Formula (C) applies; $n=16$, $r=5$, $n-r+1=12$.

$$Z = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 4368, \textit{ Ans.}$$

7. Here $$Z = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 38760, \textit{ Ans}$$

8. Omitting the boy denied the privilege of the head, the others can be arranged in

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \text{ ways.}$$

The omitted boy may occupy each of the five lower positions; hence all the ways will be

$$120 \times 5 = 600, \textit{ Ans.}$$

9. The prime numbers below 40 are 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37; or there are 13. Hence by (**341**),

$$n=13, \quad r=\frac{n-1}{2}=6, \quad n-r+1=8.$$

$$Z = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1716, \textit{ Ans.}$$

(**284**)

10. By formulas (A) and (C), we have

$$n(n-1)(n-2)(n-3)(n-4)=120\left(\frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}\right);$$

or, $\qquad n^2-7n=\ 8,$

whence, $\qquad n=\ 8,$ *Ans.*

11. Here $n=8$; and by (**341**), $r=\frac{n}{2}=4, \quad n-r+1=5.$

Hence, $\qquad x=\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}=70$; and $\frac{\$35}{70}=\$0.50$, *Ans.*

12. By formula (C), if we let $n=$ the number of horses,

$$\frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}=2\left(\frac{n(n-1)}{2\cdot 1}\right);$$ whence, $n=8$, *Ans.*

13. First omit 4 of the points which are in the same straight line; and considering the remaining 8, we find the number of combinations of 8 things taken 2 at a time to be

$$\frac{8\cdot 7}{2\cdot 1}=28;$$

or 28 different straight lines. Now the four other points may be joined each with the 7 points not in the same straight line, making 28 more different straight lines; and adding the line containing the five points, we have

$$28+28+1=57, \textit{ Ans.}$$

Again, if no three of the points are in the same straight line, there will be as many straight lines as there are combinations of 12 things, taken 2 at a time, or $\frac{12\cdot 11}{2\cdot 1}=66$. But since five of the points are in the same straight line, $\frac{5\cdot 4}{2\cdot 1}=10$ of the combinations are lost; and adding the straight line containing the five points, we have $66-10+1=57$, *Ans.*

Generally, if there are n points in a plane, of which p are in the same straight line, there will be

$$\frac{n(n-1)}{2\cdot 1}-\frac{p(p-1)}{2\cdot 1}+1$$

different straight lines formed by joining the points.

(**284**).

ARITHMETICAL PROGRESSION.

(**353**, page 288.)

NOTE.—In some of the following examples, we shall employ the formulas of 354, instead of substituting the given data in the *primary* equations.

1. $$l=7+35\cdot 3=112, \text{ Ans.}$$

2. $$s=\tfrac{46}{2}\cdot 280=6440, \text{ Ans.}$$

3. Here we have s, n and d given to find a and l. From formulas (A) and (B),

$$l-a=(n-1)d,$$

$$l+a=\frac{2s}{n}.$$ Hence,

$$a=\frac{2s-n(n-1)d}{2n};\quad l=\frac{2s+n(n-1)d}{2n};$$

and by substitution, $a=2,\ l=37$, *Ans.*

4. Substitute the value of l from formula (A), in formula (B), and we have

$$s=\frac{n}{2}\{2a+(n-1)d\},\ =\frac{101}{2}\left(2+\frac{100}{2}\right)=2626, \text{ Ans.}$$

5. By formula (C), (**352**), $d=\dfrac{37-7}{5}=6$, and the terms are

7, 13, 19, 25, 31, 37, *Ans.*

6. We have a, n and s given, to find d and l. By No. 5, (**354**),

$$l=\frac{7440}{60}-3=121;\ d=\frac{(3720-180):2}{60\cdot 59}=2, \text{ Ans.}$$

7. Here $n=11$; hence,

$$s=\tfrac{11}{2}(9+109)=649, \text{ Ans.}$$

8. By (**352**), $d=\frac{\frac{1}{2}-\frac{1}{3}}{4}=\frac{1}{24}$, *Ans.*

9. By No. 1, (**354**),

$$s=\tfrac{365}{2}\{2+(365-1)2\}=\overline{365}^2=\$1332.25, \textit{ Ans.}$$

10. By 9, (**354**), since we have given a, d and s to find n,

$$n=\frac{3-40\pm\sqrt{(37)^2+8\cdot3\cdot438}}{6}=\frac{-37\pm\sqrt{11881}}{6}=\frac{-37\pm109}{6}=12, \textit{ Ans.}$$

11. The last term will be n; and by formula (B),

$$s=\frac{n}{2}(1+n), \textit{ Ans.}$$

12. Here $a=1$, $d=2$, and $s=\frac{n}{2}\{2+(n-1)2\}=n^2$, *Ans.*

13. We have n, d, and s given; hence,

$$a=\frac{1900-24\cdot25\cdot3}{50}=\frac{1900-1800}{50}=2, \textit{ Ans.}$$

14. We have given a, d, and s to find n. By No. 9, (**354**),

$$n=\frac{\frac{1}{3}-\frac{2}{3}\pm\sqrt{\frac{1}{9}+8\cdot\frac{3775}{3}}}{\frac{2}{3}}=\frac{-\frac{1}{3}\pm\sqrt{\frac{90601}{9}}}{\frac{2}{3}};$$

or, $$n=\frac{-\frac{1}{3}\pm\frac{301}{3}}{\frac{2}{3}}=150, \textit{ Ans.}$$

15. $\frac{666-66}{120}=5=d$; and $\frac{6666-666}{5}=1200$.

Hence, $$n=1200+120+14=1334, \textit{ Ans.}$$

PROBLEMS IN ARITHMETICAL PROGRESSION.

(**355**, page 291.)

1. Let $(x-y)$, x and $(x+y)$ be the numbers.

From first condition, $3x = 18$,

$x = 6$;

from second condition, $3x^2+2y^2=158$;

whence, $2y^2 = 50$,

$y = 5$. 1, 6 and 11, *Ans.*

2. Let $(x-2y)$, $(x-y)$, x, $(x+y)$, $(x+2y)$ be the numbers.

From 1st condition, $5x = 65$,

$x = 13$;

" 2d " $5x^2+10y^2=1005$,

$x^2+2y^2= 201$;

whence, $2y^2= 32$,

$y = 4$. 5, 9, 13, 17, 21, *Ans.*

3. Let $(x-6)$, $(x-2)$, $(x+2)$ and $(x+6)$ be the numbers.

$$(x^2-4)(x^2-36)=x^4-40x^2+144=176985,$$

or,
$$x^4-40x^2=176841,$$
$$x^4-40x^2+400=177241,$$
$$x^2=20\pm421,$$
$$x^2=441,$$
$$x=21$$

15, 19, 23, 27, *Ans.*

4. Let $(x-3y)$, $(x-y)$, $(x+y)$ and $(x+3y)$ be the numbers.

$2x= 8$,

or, $x= 4$;

$x^2-y^2=15$;

whence, $y^2= 1$,

$y = 1$. 1, 3, 5, 7, *Ans.*

5. Let $n=$ the number of days the first person travels,

$$d=1, \qquad \text{and } l=1+(n-1)d=n.$$

$$s=\frac{n}{2}(1+n)=\text{ the whole distance,}$$

and $\qquad 15(n-6)=$ " "

hence, $\qquad \frac{1}{2}n(1+n)=15(n-6),$

or, $\qquad n^2+n=30n-180,$

$$n^2-29n=-180.$$

First person travels, $\qquad n=\frac{29}{2}\pm\frac{11}{2}=\quad 9 \text{ or } 20,$

$$-6 \quad -6,$$

Second " " $\qquad 3$ or 14, *Ans.*

EXPLANATION.—Call the first person A, and the second B. Now B overtakes and passes A after A has traveled 9 days and B 3 days. But as A is increasing his rate one mile per day, he finally gains on B, and overtakes and passes him after A has traveled 20 days, and B 14. They are together after having traveled 45 and 201 miles.

6. Let $x=$ one of the equal payments.

At the given rate per cent. \$60 will amount to \$61 at the end of 60 days. As the rate of interest is $\frac{1}{3600}$ of the principal for a day, the first partial payment will amount to

$$x+\tfrac{59}{3600}x,$$

the second to $\qquad x+\frac{58}{3600}x,$

the third to $\qquad x+\frac{57}{3600}x,$

and the last to $\qquad x.$

Hence the sum of the partial payments is,

$$60x+\tfrac{1}{3600}(59x+58x+57x\ldots.+x),$$

or by summing the series,

$$60x+\tfrac{59}{120}x.$$

Since the debt is to be canceled, we must have

$$60x+\tfrac{59}{120}x=\quad 61,$$

$$7259x=7320,$$

whence $\qquad x=\$1\tfrac{61}{7259}$, *Ans.*

7. Let $(x-3y$, $(x-y)$, $(x+y)$, and $(x+3y)$ be the numbers; then

$$2x^2+18y^2=65, \qquad (1)$$
$$2x^2+2y^2=61, \qquad (2)$$
$$y^2=\tfrac{1}{4},$$
$$y=\tfrac{1}{2},$$
$$x=\tfrac{11}{2}; \qquad 4,\ 5,\ 6,\ 7,\ \textit{Ans.}$$

8. Adopting the same notation as before,

$$4x=24, \qquad (1)$$
$$x^4-10x^2y^2+9y^4=945. \qquad (2)$$

Whence, $\quad x=6;$

and from (2), $\quad y^4-40y^2=-39,$

$$y^2=\ \ 20\pm19,$$
$$y=1, \qquad 3,\ 5,\ 7,\ 9,\ \textit{Ans.}$$

9. Let $(x-y)$, x, and $(x+y)$ be the digits. We have

$$\frac{100(x-y)+10x+x+y}{3x}=26, \qquad (1)$$

$$100(x-y)+10x+x+y+198=100(x+y)+10x+x-y; \qquad (2)$$

from the second equation, $\quad y=1,$

from the first, $\quad 3x=9y,$

$$x=3; \qquad 234,\ \textit{Ans.}$$

10. Let $n=$ the number of days. Then from No. 2, (**354**),

$$\frac{n}{2}\{6+(n-1)2\}=\text{ the distance A travels,}$$

$$\frac{n}{2}\{8+(n-1)2\}=\quad \text{“} \quad \text{B} \quad \text{“}$$

The sum of these expressions is equal to the whole distance; hence,

$$2n^2+5n=102,$$
$$2n=-\tfrac{5}{2}\pm\tfrac{29}{2}; \qquad n=6,\ \textit{Ans.}$$

11. Let $x=$ the number of weeks if no one dies;

$21x=$ the pecks of corn distributed.

In the second case the number of pecks distributed each week will form an arithmetical series, in which

$$a=21, \quad d=-1, \quad n=2x;$$

and

$$S=x(42-2x+1)=\text{ the pecks of corn.}$$

Equating the two expressions for the number of pecks, we obtain

$$43x-2x^2=21x,$$
$$2x=22,$$
$$x=11; \qquad 21x=231, \textit{ Ans.}$$

GEOMETRICAL PROGRESSION.

(**364**, page 296.)

1. Here $a=1, \qquad r=2.$

$$S=\frac{2^9-1}{2-1}=\frac{512-1}{1}=511, \textit{ Ans.}$$

2. $a=2, \qquad r=3$; hence,

$$l=2\cdot 3^7=2\cdot 2187=4374, \textit{ Ans.}$$

3. $a=1, \qquad r=\frac{2}{3}$; hence,

$$S=\frac{(\frac{2}{3})^{10}-1}{\frac{2}{3}-1}=\frac{\frac{1024}{59049}-4}{\frac{2}{3}-1}=\frac{58025}{59049}\cdot 3=\frac{174075}{59049}, \textit{ Ans.}$$

4. $r=(\frac{192}{24})^{\frac{1}{3}}=2.$ Hence, 48, 96, *Ans.*

5. $r=(\frac{768}{3})^{\frac{1}{8}}=(256)^{\frac{1}{8}}=2.$ Hence,

6, 12, 24, 48, 96, 192, *Ans.*

6. $a=1, \qquad r=\frac{3}{4}$; hence,

$$S=\frac{1}{1-\frac{3}{4}}=4, \textit{ Ans.}$$

7. $a=\frac{5}{3}, \qquad r=\frac{3}{5}$; hence,

$$S=\frac{\frac{5}{3}}{1-\frac{3}{5}}=\frac{5}{3}\cdot\frac{5}{2}=4\tfrac{1}{6}, \textit{ Ans.}$$

8. $a=5, \quad r=\frac{1}{3}$; hence,

$$S=\frac{5}{1-\frac{1}{3}}=5\cdot\frac{3}{2}=7\tfrac{1}{2}, \text{ Ans.}$$

9. $a=\frac{32}{100}, \quad r=\frac{1}{100}$; hence,

$$S=\frac{\frac{32}{100}}{1-\frac{1}{100}}=\frac{32}{100}\cdot\frac{100}{99}=\frac{32}{99}, \text{ Ans.}$$

10. $a=\frac{21}{100}, \quad r=\frac{1}{100}$; hence,

$$S=\frac{\frac{21}{100}}{1-\frac{1}{100}}=\frac{21}{100}\cdot\frac{100}{99}=\frac{7}{33}, \text{ Ans.}$$

11. $a=\frac{1}{2}, \quad r=-\frac{1}{2}$; hence,

$$S=\frac{\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}, \text{ Ans.}$$

12. $a=\frac{1}{5}, \quad r=-\frac{1}{5}$; hence,

$$S=\frac{\frac{1}{5}}{1+\frac{1}{5}}=\frac{1}{5}\cdot\frac{5}{6}=\frac{1}{6}, \text{ Ans.}$$

13. $a=1, \quad r=\frac{x}{a}$; hence,

$$S=\frac{1}{1-\frac{x}{a}}=\frac{1}{\frac{a-x}{a}}=\frac{a}{a-x}, \text{ Ans.}$$

14. $a=\frac{1}{a}, \quad r=-\frac{x^2}{a^2}$; hence,

$$\frac{\frac{1}{a}}{1+\frac{x^2}{a^2}}=\frac{1}{a}\cdot\frac{a^2}{a^2+x^2}=\frac{a}{a^2+x^2}, \text{ Ans.}$$

15. By formula (B),

$$1785=\frac{a(2^8-1)}{2-1}=255a. \quad \text{Hence,} \quad a=7, \text{ Ans.}$$

16. By formula (B'),

$$7812 = \frac{5l + a}{5 - 1},$$

or,
$$l = \frac{31248 + a}{5}.$$

By (B),
$$7812 = \frac{a(5^6 - 1)}{5 - 1} = 3906a.$$

Hence, $a = 2$, and $l = 6250$, *Ans.*

17. By formula (A),

$$1215 = 5r^5,$$

or, $r^5 = 243$. Hence, $r = 3$, *Ans.*

18. By formula (B),

$$S = \frac{2^{10} - 1}{2 - 1} = \frac{1024 - 1}{1} = 1023, \textit{ Ans.}$$

PROBLEMS IN GEOMETRICAL PROGRESSION.

(**365**, page 300.)

3. Let the numbers be x, $\sqrt{xy}$, y; then

$$x + \sqrt{xy} + y = 21, \qquad (1)$$
$$x^2 + xy + y^2 = 189, \qquad (2)$$

The solution is like that of Example 1. We have $a = 21$, $b = 189$.

$$\sqrt{xy} = \frac{441 - 189}{42} = 6;$$

and from (1) and (2), $x + y = 15$,
$x - y = 9$. Hence, 3, 6, 12, *Ans.*

4. Let x, xy, xy^2 be the numbers; then

$$x + xy + xy^2 = 210, \qquad (1)$$
$$xy^2 - x = 90. \qquad (2)$$

Dividing by x, and eliminating y^2, we have

$$x=\frac{120}{2+y}.$$

Substituting this value of x in (2),

$$\begin{aligned} 12y^2-9y&=30, \\ y&=2, \\ x&=30; \qquad 30,\ 60,\ 120,\ \textit{Ans.} \end{aligned}$$

5. Let x, xy, xy^2, xy^3 be the numbers; then

$$x+xy+xy^2+xy^3=30, \qquad (1)$$

$$\frac{xy^3}{xy+xy^2}=\frac{4}{3}. \qquad (2)$$

From (2), $$\frac{y^2}{1+y}=\frac{4}{3},$$

or, $$y=2;$$

whence by (1), $$x+2x+4x+8x=30,$$

$$x=2; \qquad 2,\ 4,\ 8,\ 16,\ \textit{Ans.}$$

6. Adopt the same notation as before; then,

$$x+xy^2=148, \qquad (1)$$

$$xy+xy^3=888. \qquad (2)$$

Dividing (2) by (1), $$y=6,$$

from (1), $$37x=148,$$

$$x=4; \qquad 4,\ 24,\ 144,\ 864,\ \textit{Ans.}$$

7. Solution the same as in 1 and 3.

8. Let x, xy, xy^2, xy^3 be the numbers; then,

$$xy^2-xy=24; \qquad (1)$$

$$x+xy^3 : xy+xy^2 = 7:3,$$

or, $$1+y^3 : y+y^2 = 7:3,$$

dividing by $y+1$, $$y^2-y+1 : y = 7:3,$$

$$3y^2-10y=-3; \qquad (2)$$

from (2), $$y=3,$$

whence from (1), $$x=1; \qquad 1,\ 3,\ 9,\ 27,\ \textit{Ans.}$$

9. Let x, xy, xy^2 be the numbers; then,

$$x+xy=20, \qquad (1)$$
$$xy^2-xy=30. \qquad (2)$$

By division,
$$\frac{1+y}{y^2-y}=\frac{2}{3},$$

or,
$$2y^2-5y=3;$$
$$y=3,$$
$$x=5; \qquad 5, 15, 45, \textit{Ans.}$$

10. Adopt the same notation as before; then,

$$x^3y^3=216, \qquad (1)$$
$$x^2+x^2y^4=328. \qquad (2)$$

From (1),
$$xy=6,$$
$$x^2=\frac{36}{y^2};$$

from (2),
$$x^2=\frac{328}{1+y^4};$$

hence,
$$9y^4-82y^2=-9;$$
$$y=3,$$
$$x=2; \qquad 2, 6, 18, \textit{Ans.}$$

11. Let $x, \sqrt{xy}, y$ be the numbers; then

$$x+\sqrt{xy}+y=13, \qquad (1)$$
$$(x+y)\sqrt{xy}=30. \qquad (2)$$

Hence,
$$x+y=13-\sqrt{xy},$$
$$x+y=\frac{30}{\sqrt{xy}};$$
$$xy-13\sqrt{xy}=-30,$$
$$\sqrt{xy}=3,$$
$$xy=9,$$
$$x+y=10,$$
$$x-y=8; \qquad 1, 3, 9, \textit{Ans.}$$

12. Let x, xy, xy^2, be the numbers; then

$$x^3y^3 = 64, \qquad (1)$$

$$x^3+x^3y^3+x^3y^6=584. \qquad (2)$$

From (1), $$x^3=\frac{64}{y^3};$$

by substitution in (2), $$16y^6-130y^3=-16,$$

$$y=2,$$

whence, $$x=2; \qquad 2, 4, 8, \textit{Ans.}$$

13. Let x, xy, xy^2 be the numbers; then

$$x^3y^3=1; \qquad (1)$$

and, $$xy-x : xy^2-xy=5 : 1,$$

or, $$y-1 : y^2-y=5 : 1,$$

or, $$1 : y=5 : 1,$$

whence, $$y=\tfrac{1}{5}; \qquad (2)$$

from (1), $$xy=1,$$

$$x=5; \qquad 5, 1, \tfrac{1}{5}, \textit{Ans.}$$

14. We might divide 120 into four parts, which should be in arithmetical progression, by assuming a *single* unknown quantity for the first term, or the common difference; but as we have two conditions to satisfy, we take *two* unknown quantities. Let x be the first term, and y the common difference; then

$$x+(x+y)+(x+2y)+(x+3y)=120,$$

or, $$2x+3y=60. \qquad (1)$$

Again, we have by the second condition,

$$x,\ (x+y-12),\ (x+2y-12),\ (x+3y+24),$$

for the geometrical series. Hence, by (**363**),

$$(x+y-12)^2=x(x+2y-12), \qquad (2)$$

or, $$y^2-12x-24y=-144;$$

substituting from (1), $$y^2-6y=216,$$

$$y=18,$$

$$x=3; \qquad 3, 21, 39, 57, \textit{Ans.}$$

15. Let x, $\sqrt{xy}$, y, be the numbers; then

$$x+\sqrt{xy}+y=31, \qquad (1)$$
$$x+y=26, \qquad (2)$$
$$\sqrt{xy}=5,$$
$$xy=25,$$

whence,

$$x-y=24,$$
$$x+y=26; \qquad 1, 5, 25, \textit{Ans.}$$

16. Let x, xy, xy^2, xy^3, xy^4, xy^5, be the numbers; then

$$x+xy+xy^2+xy^3+xy^4+xy^5=189, \qquad (1)$$
$$xy+xy^4=54, \qquad (2)$$

from (1), $\quad x(1+y+y^2)+xy^3(1+y+y^2)=189,$

$$x+xy^3=\frac{189}{1+y+y^2},$$

Multiplying this equation by y, and equating with (2),

$$54=\frac{189y}{1+y+y^2},$$
$$2y^2-5y=-2,$$
$$y=2,$$
$$x=3;$$
$$3, 6, 12, 24, 48, 96, \textit{Ans.}$$

17. Adopting the same notation as before, we have

$$(x+xy)+(x+xy)y^4=189-36=153, \qquad (1)$$
$$(x+xy)y^2=36. \qquad (2)$$

By division, $\quad \dfrac{1+y^4}{y^2}=\dfrac{17}{4},$

whence, $\quad y=2,$

$$x=3; \quad 3, 6, 12, 24, 48, 96, \textit{Ans.}$$

18. Let x represent one of the equal payments; then

$p(1+r)-x,$	due after	1st	payment,
$p(1+r)^2-x(1+r)-x,$	"	2d	"
$p(1+r)^3-x(1+r)^2-x(1+r)-x,$	"	3d	"
etc., etc., etc.,			

or, $$p(1+r)^n-x(1+r)^{n-1}-\ldots\ldots\ldots-x(1+r)-x,$$

due after the nth payment. And since the debt is to be canceled by the nth, or last payment, we have

$$p(1+r)^n-x(1+r)^{n-1}-x(1+r)^{n-2}-\ldots\ldots-x(1+r)-x=0\,;$$

or, $$p(1+r)^n-\{(1+r)^{n-1}+(1+r)^{n-2}+\ldots\ldots+(1+r)+1\}x=0.$$

Summing the series in the parenthesis,

$$p(1+r)^n-\left(\frac{(1+r)^n-1}{r}\right)x=0\,;$$

whence, $$r=\frac{pr(1+r)^n}{(1+r)^n-1},\ \textit{Ans.}$$

DECOMPOSITION OF RATIONAL FRACTIONS.

(**369**, page 308.)

1. $$x^2-9x+14=(x-7)(x-2).$$

Hence we assume,

$$\frac{7x-24}{(x-7)(x-2)}=\frac{A}{x-7}+\frac{B}{x-2}\,;$$

whence, $$7x-24=(A+B)x-(2A+7B).$$

Equating coefficients,

$$A+B=7,$$
$$2A+7B=24\,;$$

whence, $$A=5,\quad B=2.$$

$$\frac{5}{x-7}+\frac{2}{x-2},\ \textit{Ans.}$$

2. $$2x^2+3x-20=(2x-5)(x+4).$$

Assume $$\frac{20x+2}{(2x-5)(x+4)}=\frac{A}{2x-5}+\frac{B}{x+4}\,;$$

then $$20x+2=(A+2B)x+(4A-5B).$$

Equating coefficients,

$$A+2B=20,$$
$$4A-5B=2;$$

whence, $$A=8, \quad B=6.$$

$$\frac{8}{2x-5}+\frac{6}{x+4}, \text{ Ans.}$$

3. Assume

$$\frac{6x^2-22+18}{(x-1)(x^2-5x+6)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3};$$

then

$$6x^2-22x+18=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2).$$

Hence we have

$$A+B+C=6,$$
$$5A+4B+3C=22,$$
$$6A+3B+2C=18.$$

Whence, $$A=1, \quad B=2, \quad C=3.$$

$$\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}, \text{ Ans.}$$

4. Let $$\frac{x+2}{x^3-x}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1};$$

then $$x+2=A(x^2-1)+B(x^2-x)+C(x^2+x).$$

Thence we have

$$A+B+C=0,$$
$$-B+C=1,$$
$$-A=2.$$

Whence, $$A=-2, \quad B=\tfrac{1}{2}, \quad C=\tfrac{3}{2}.$$

$$-\frac{2}{x}+\frac{1}{2(x+1)}+\frac{3}{2(x-1)}, \text{ Ans.}$$

5. **We have**

$$x^4-13x^2+36=(x^2-4)(x^2-9)=(x+2)(x-2)(x+3)(x-3).$$

Hence we assume

$$\frac{10}{x^4-13x^2+36}=\frac{A}{x+2}+\frac{B}{x-2}+\frac{C}{x+3}+\frac{D}{x-3};$$

from which we find

$$\begin{aligned}10= \ & A(x^3-2x^2-9x+18)+B(x^3+2x^2-9x-18)\\ &+C(x^3-3x^2-4x+12)+D(x^3+3x^2-4x-12).\end{aligned}$$

And equating coefficients,

$$\begin{aligned}A+B+C+D&=0,\\ -2A+2B-3C+3D&=0,\\ 9A+9B+4C+4D&=0,\\ 18A-18B+12C-12D&=10.\end{aligned}$$

Hence, combining the first and third equations,

$$A=-B, \quad \text{and } C=-D.$$

From the second and fourth equations,

$$\begin{aligned}2A-3D&=0,\\ 18A-12D&=5.\end{aligned}$$

Hence, $\quad A=\frac{1}{2}, \quad B=-\frac{1}{2}, \quad C=-\frac{1}{3}, \quad D=\frac{1}{3}.$

$$\frac{1}{2(x+2)}-\frac{1}{2(x-2)}-\frac{1}{3(x+3)}+\frac{1}{3(x-3)}, \textit{ Ans.}$$

BINOMIAL THEOREM.

(**377**, page 316.)

In examples 1 to 10 the exponents are whole numbers, so that these examples need no solution here. In example 10 the numerical coefficients are the same as those of example 2, of the illustrations; and we shall have the answer by simply changing $+x$ to $-x$ and observing that the odd powers of $-x$ are negative.

(308—316)

11. Here we have

$$\begin{aligned}
&A && = +1\\
&B = A\times n && = +\tfrac{1}{3}\\
&C = B\times\frac{n-1}{2} = -\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{2} && = -\frac{2}{3\cdot 6}\\
&D = C\times\frac{n-2}{3} = +\frac{2}{3\cdot 6}\cdot\frac{5}{3}\cdot\frac{1}{3} && = +\frac{2\cdot 5}{3\cdot 6\cdot 9}\\
&E = D\times\frac{n-3}{4} = -\frac{2\cdot 5}{3\cdot 6\cdot 9}\cdot\frac{8}{3}\cdot\frac{1}{4} && = -\frac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}
\end{aligned}$$

The law of the numerical coefficients is now evident, and we may write out the series, observing that the odd powers of $-x$ are negative. Hence,

$$(1-x)^{\frac{1}{3}}=1-\frac{x}{3}-\frac{2x^2}{3\cdot 6}-\frac{2\cdot 5x^3}{2\cdot 6\cdot 9}-\frac{2\cdot 5\cdot 8x^4}{3\cdot 6\cdot 9\cdot 12}-\frac{2\cdot 5\cdot 8\cdot 11x^5}{3\cdot 6\cdot 9\cdot 12\cdot 15}-\ldots\ldots,$$

Ans.

12.

$$\begin{aligned}
&A && = +1\\
&B = A\times n && = +\tfrac{1}{4}\\
&C = B\times\frac{n-1}{2} = -\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2} && = -\frac{3}{4\cdot 8}\\
&D = C\times\frac{n-2}{3} = +\frac{3}{4\cdot 8}\cdot\frac{7}{4}\cdot\frac{1}{3} && = +\frac{3\cdot 7}{4\cdot 8\cdot 12}\\
&E = D\times\frac{n-3}{4} = -\frac{3\cdot 7}{4\cdot 8\cdot 12}\cdot\frac{11}{4}\cdot\frac{1}{4} && = -\frac{3\cdot 7\cdot 11}{4\cdot 8\cdot 12\cdot 16}
\end{aligned}$$

Hence,

$$(a+1)^{\frac{1}{4}}=a^{\frac{1}{4}}+\frac{a^{-\frac{3}{4}}}{4}-\frac{3a^{-\frac{7}{4}}}{4\cdot 8}+\frac{3\cdot 7a^{-\frac{11}{4}}}{4\cdot 8\cdot 12}-\frac{3\cdot 7\cdot 11a^{-\frac{15}{4}}}{4\cdot 8\cdot 12\cdot 16}+\ldots\ldots$$

$$=a^{\frac{1}{4}}\left(1+\frac{1}{4a}-\frac{2}{4\cdot 8a^2}+\frac{3\cdot 7}{4\cdot 8\cdot 12a^3}-\frac{3\cdot 7\cdot 11}{4\cdot 8\cdot 12\cdot 16a^4}+\ldots\ldots\right),$$

Ans.

13. Since n is the same as in Example 11, the numerical coefficients will be the same. Hence,

$$(a+b)^{\frac{1}{3}}=a^{\frac{1}{3}}+\frac{a^{-\frac{2}{3}}b}{3}-\frac{2\cdot a^{-\frac{5}{3}}b^2}{3\cdot 6}+\frac{2\cdot 5a^{-\frac{8}{3}}b^3}{3\cdot 6\cdot 9}-\frac{2\cdot 5\cdot 8a^{-\frac{11}{3}}b^4}{3\cdot 6\cdot 9\cdot 12}+\ldots\ldots$$

$$=a^{\frac{1}{3}}\left(1+\frac{b}{2a}-\frac{2b^2}{3\cdot 6a^2}+\frac{2\cdot 5b^3}{3\cdot 6\cdot 9a^3}-\frac{2\cdot 5\,.\,8b^4}{3\cdot 6\cdot 9\cdot 12a^4}+\ldots\ldots\right),\ \textit{Ans.}$$

14. $$\frac{1}{a-b}=(a-b)^{-1};$$

$$\begin{aligned}
&A &&= +1\\
&B = A \times n &&= -1\\
&C = B \times \frac{n-1}{2} = (-1)\times(-1) &&= +1\\
&D = C \times \frac{n-2}{3} = (+1)\times(-1) &&= -1\\
&E = D \times \frac{n-3}{4} = (-1)\times(-1) &&= +1
\end{aligned}$$

Hence, $$(a-b)^{-1}=a^{-1}+a^{-2}b+a^{-3}b^2+a^{-4}b^3+\ldots.$$
$$=\frac{1}{a}+\frac{b}{a^2}+\frac{b^2}{a^3}+\frac{b^3}{a^4}+\ldots., \textit{Ans.}$$

15. $$\frac{a}{(1-x)^2}=a(1-x)^{-2}.$$ Omitting the factor a,

$$\begin{aligned}
&A &&=+1\\
&B = A \times n = (+1)\times(-2) &&=-2\\
&C = B \times \frac{n-1}{2} = (-2)\times(-\tfrac{3}{2}) &&=+3\\
&D = C \times \frac{n-2}{3} = (+3)\times(-\tfrac{4}{3}) &&=-4\\
&E = D \times \frac{n-3}{4} = (-4)\times(-\tfrac{5}{4}) &&=+5.
\end{aligned}$$
Hence,

$$a(1-x)^{-2}=a(1+2x+3x^2+4x^3+5x^4+6x^5+\ldots.), \textit{Ans.}$$

16. The coefficients are the same as in Example 10.

17.
$$\begin{aligned}
&A &&=+1\\
&B = A\times n = (+1)\cdot(+\tfrac{2}{3}) &&=+\tfrac{2}{3}\\
&C = B\times\frac{n-1}{2} = \left(+\frac{2}{3}\right)\cdot\left(-\frac{1}{3}\right)\cdot\left(\frac{1}{2}\right) &&=-\frac{2}{3\cdot 6}\\
&D = C\times\frac{n-2}{3} = \left(-\frac{2}{3\cdot 6}\right)\cdot\left(-\frac{4}{3}\right)\cdot\left(\frac{1}{3}\right) &&=+\frac{2\cdot 4}{3\cdot 6\cdot 9}
\end{aligned}$$

$$E = D \times \frac{n-3}{4} = \left(+\frac{2 \cdot 4}{3 \cdot 6 \cdot 9}\right) \cdot \left(-\frac{7}{3}\right) \cdot \left(\frac{1}{4}\right) = -\frac{2 . 4 . 7}{3 \cdot 6 \cdot 9 \cdot 12}$$

Hence,

$$(a-c^2)^{\frac{2}{3}} = a^{\frac{2}{3}} - \frac{2a^{-\frac{1}{3}}c^2}{3} - \frac{2a^{-\frac{4}{3}}c^4}{3 \cdot 6} - \frac{2 \cdot 4a^{-\frac{7}{3}}c^6}{3 \cdot 6 \cdot 9} - \frac{2 \cdot 4 \cdot 7a^{-\frac{10}{3}}c^8}{3 \cdot 6 \cdot 9 \cdot 12} - \ldots.$$

$$= a^{\frac{2}{3}}\left(1 - \frac{2c^2}{3a} - \frac{2c^4}{3 \cdot 6a^2} - \frac{2 \cdot 4c^6}{3 \cdot 6 \cdot 9a^3} - \frac{2 \cdot 4 \cdot 7c^8}{3 \cdot 6 \cdot 9 \cdot 12a^4} - \ldots.\right), \text{ *Ans.*}$$

18.
$$\begin{aligned}
A &= & &= +1 \\
B &= A \times n &= (+1) \cdot (-\tfrac{1}{2}) & = -\tfrac{1}{2} \\
C &= B \times \frac{n-1}{2} &= \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdot \left(\frac{1}{2}\right) &= +\frac{3}{2 \cdot 4} \\
D &= C \times \frac{n-2}{3} &= \left(+\frac{3}{2 \cdot 4}\right) \cdot \left(-\frac{5}{2}\right) \cdot \left(\frac{1}{3}\right) &= -\frac{3 \cdot 5}{2 \cdot 4 \cdot 6} \\
E &= D \times \frac{n-3}{4} &= \left(-\frac{3 \cdot 5}{2 \cdot 4 \cdot 6}\right) \cdot \left(-\frac{7}{2}\right) \cdot \left(\frac{1}{4}\right) &= +\frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}
\end{aligned}$$

Hence, we have

$$d(c^2+x^2)^{-\frac{1}{2}} = d\left(c^{-1} - \frac{c^{-3}x^2}{2} + \frac{3c^{-5}x^4}{2 \cdot 4} - \frac{3 \cdot 5c^{-7}x^6}{2 \cdot 4 \cdot 6} + \frac{3 \cdot 5 \cdot 7c^{-9}x^8}{2 \cdot 4 \cdot 6 \cdot 8} - \ldots.\right)$$

$$= \frac{d}{c}\left(1 - \frac{x^2}{2c^2} + \frac{3x^4}{2 \cdot 4c^4} - \frac{3 \cdot 5x^6}{2 \cdot 4 \cdot 6c^6} + \frac{3 \cdot 5 \cdot 7x^8}{2 \cdot 4 \cdot 6 \cdot 8c^8} - \ldots.\right), \text{ *Ans.*}$$

NOTE.—The exponent of c in the first term is $(c^2)^{-\frac{1}{2}} = c^{-1}$; and the exponents of c in the terms following diminish by 2 throughout.

19.
$$\begin{aligned}
A & & &= +1 \\
B &= A \times n & &= -3 \\
C &= B \times \frac{n-1}{2} &= (-3) \times \left(-\frac{4}{2}\right) &= +6 \\
D &= C \times \frac{n-2}{3} &= (+6) \times \left(-\frac{5}{3}\right) &= -10 \\
E &= D \times \frac{n-3}{4} &= (-10) \times \left(-\frac{6}{4}\right) &= +15
\end{aligned}$$

Hence the law of the coefficients is evident, and we have

$$(1-a)^{-3} = 1 + 3a + 6a^2 + 10a^3 + 15a^4 + 21a^5 + 28a^6 + \ldots., \text{ *Ans.*}$$

20. $$\begin{aligned} &A & & &&= +1 \\ &B = A \times n &&= (+1)\cdot\tfrac{3}{4} &&= +\tfrac{3}{4} \\ &C = B\times\frac{n-1}{2} &&= \left(+\frac{3}{4}\right)\cdot\left(-\frac{1}{4}\right)\cdot\left(\frac{1}{2}\right) &&= -\frac{3}{4\cdot 8} \\ &D = C\times\frac{n-2}{3} &&= \left(-\frac{3}{4\cdot 8}\right)\cdot\left(-\frac{5}{4}\right)\cdot\left(\frac{1}{3}\right) &&= +\frac{3\cdot 5}{4\cdot 8\cdot 12} \\ &E = D\times\frac{n-3}{4} &&= \left(+\frac{3.5}{4\cdot 8\cdot 12}\right)\cdot\left(-\frac{9}{4}\right)\cdot\left(\frac{1}{4}\right) &&= -\frac{3\cdot 5\cdot 9}{4\cdot 8\cdot 12\cdot 16} \end{aligned}$$

Hence,

$$(a^2-x^2)^{\frac{3}{4}} = a^{\frac{3}{2}} - \frac{3a^{-\frac{1}{2}}x^2}{4} - \frac{3a^{-\frac{5}{2}}x^4}{4\cdot 8} - \frac{3\cdot 5a^{-\frac{9}{2}}x^6}{4.8\cdot 12} - \frac{3\cdot 5\cdot 9a^{-\frac{13}{2}}x^8}{4\cdot 8\cdot 12\cdot 16} - \ldots\ldots$$

$$= \sqrt{a}\left(a - \frac{3x^2}{3a} - \frac{3x^4}{4\cdot 8a^3} - \frac{3\cdot 5x^6}{4\cdot 8\cdot 12a^5} - \frac{3\cdot 5\cdot 9x^8}{4\cdot 3\cdot 12\cdot 16a^7} - \ldots\right),$$

Ans.

NOTE.—The exponent of a in the first term is $(a^2)^{\frac{3}{4}} = a^{\frac{3}{2}}$; the exponents in the terms following diminish by 2 throughout.

21. $$\begin{aligned} &A & & &&= +1 \\ &B = A\times n &&= (+1)\times(-4) &&= -4 \\ &C = B\times\frac{n-1}{2} &&= (-4)\times\left(-\frac{5}{2}\right) &&= +10 \\ &D = C\times\frac{n-2}{3} &&= (+10)\times\left(-\frac{6}{3}\right) &&= -20 \\ &E = D\times\frac{n-3}{4} &&= (-20)\times\left(-\frac{7}{4}\right) &&= +35 \\ &F = E\times\frac{n-4}{5} &&= (+35)\times\left(-\frac{8}{5}\right) &&= -56 \end{aligned}$$

Hence,

$$(a+y)^{-4} = a^{-4} - 4a^{-5}y + 10a^{-6}y^2 - 20a^{-7}y^3 + 35a^{-8}y^4 - 56a^{-9}y^5 + \ldots\ldots$$

$$= \frac{1}{a^4} - \frac{4y}{a^5} + \frac{10y^2}{a^6} - \frac{20y^3}{a^7} + \frac{36y^4}{a^8} - \frac{56y^5}{a^9} + \ldots\ldots, \text{ Ans.}$$

22. We have $\dfrac{r}{\sqrt[5]{1-r}} = r(1-r)^{-\frac{1}{5}};$

$$\begin{aligned}
&A && && = +1\\
&B = A \times n && = (+1)\cdot\left(-\tfrac{1}{5}\right) && = -\tfrac{1}{5}\\
&C = B \times \frac{n-1}{2} && = \left(-\frac{1}{5}\right)\cdot\left(-\frac{6}{5}\right)\cdot\left(\frac{1}{2}\right) && = +\frac{6}{2\cdot 5^2}\\
&D = C \times \frac{n-2}{3} && = \left(+\frac{6}{2.5^2}\right)\cdot\left(-\frac{11}{5}\right)\cdot\left(\frac{1}{3}\right) && = -\frac{6\cdot 11}{2\cdot 3\cdot 5^3}.\\
&E = D \times \frac{n-3}{4} && = \left(-\frac{6\cdot 11}{2\cdot 3\cdot 5^3}\right)\cdot\left(-\frac{16}{5}\right)\cdot\left(\frac{1}{4}\right) && = +\frac{6\cdot 11\cdot 16}{2\cdot 3\cdot 4\cdot 5^4}
\end{aligned}$$

Hence,

$$r(1-r)^{-\frac{1}{5}} = r\left(1+\frac{r}{5}+\frac{6r^2}{2\cdot 5^2}+\frac{6\cdot 11r^3}{2\cdot 3\cdot 5^3}+\frac{6\cdot 11\cdot 16r^4}{2\cdot 3\cdot 4\cdot 5^4}+\cdots\cdot\right.$$

$$= r+\frac{r^2}{5}+\frac{6r^3}{2\cdot 5^2}+\frac{6\cdot 11r^4}{2\cdot 3\cdot 5^3}+\frac{6\cdot 11\cdot 16r^5}{2\cdot 3\cdot 4\cdot 5^4}+\cdots\cdot, \textit{ Ans.}$$

23. $\sqrt[15]{1-x^4} = (1-x^4)^{\frac{1}{15}}.$

$$\begin{aligned}
&A && && = +1\\
&B = A \times n && = (+1)\cdot\left(\tfrac{1}{15}\right) && = +\tfrac{1}{15}\\
&C = B \times \frac{n-1}{2} && = \left(+\frac{1}{15}\right)\cdot\left(-\frac{14}{15}\right)\cdot\left(\frac{1}{2}\right) && = -\frac{14}{2\cdot 15^2}\\
&D = C \times \frac{n-2}{3} && = \left(-\frac{14}{2\cdot 15^2}\right)\cdot\left(\frac{29}{15}\right)\cdot\left(\frac{1}{3}\right) && = +\frac{14\cdot 29}{2\cdot 3\cdot 15^3}\\
&E = D \times \frac{n-3}{4} && = \left(+\frac{14\cdot 29}{2\cdot 3\cdot 15^3}\right)\cdot\left(-\frac{44}{15}\right)\cdot\left(\frac{1}{4}\right) && = -\frac{14\cdot 29\cdot 44}{2\cdot 3\cdot 4\cdot 15^4}
\end{aligned}$$

Hence,

$$\sqrt[15]{1-x^4} = 1-\frac{x^4}{15}-\frac{14x^8}{2\cdot 15^2}-\frac{14\cdot 29x^{12}}{2\cdot 3\cdot 15^3}-\frac{14\cdot 29\cdot 44x^{16}}{2\cdot 3\cdot 4\cdot 15^4}-\cdots\cdot, \textit{ Ans.}$$

In all expansions of this kind, the chief thing to be aimed at, is a simple and systematic method of calculating the numerical coefficients, and one which will clearly show the law of their formation.

(**378**, page 318.)

1. $(a-2b)^3=a^3-3a^2(2b)+3a(2b)^2+(2b)^3$
$$=a^3-6a^2b+12ab^2+8b^3, \textit{ Ans.}$$

2. $(2a+3x)^4=(2a)^4+4(2a)^3(3x)+6(2a)^2(3x)^2+4(2a)(3x)^3+(3x)^4$
$$=16a^4+96a^3x+216a^2x^2+216ax^3+81x^4, \textit{ Ans.}$$

3. $\left(1-\frac{a}{2}\right)^4=1-4\left(\frac{a}{2}\right)+6\left(\frac{a}{2}\right)^2-4\left(\frac{a}{2}\right)^3+\left(\frac{a}{2}\right)^4$
$$=1-2a+\tfrac{3}{2}a^2-\tfrac{1}{2}a^3+\tfrac{1}{16}a^4, \textit{ Ans.}$$

4. $(a^2-ax+x^2)^4=(a^2-ax)^4+4(a^2-ax)^3x^2+6(a^2-ax)^2x^4+4(a^2-ax)x^6+x^8.$

Performing the operations indicated,

$$\begin{array}{l}
a^8-4a^7x+6a^6x^2-4a^5x^3+a^4x^4 \\
\qquad +4a^6x^2-12a^5x^3+12a^4x^4-4a^3x^5 \\
\qquad\qquad +6a^4x^4-12a^3x^5+6a^2x^6 \\
\qquad\qquad\qquad +4a^2x^6-4ax^7+x^8 \\
\hline
a^8-4a^7x+10a^6x^2-16a^5x^3+19a^4x^4-16a^3x^5+10a^2x^6-4ax^7+x^8, \\
\hfill \textit{Ans.}
\end{array}$$

5. The numerical coefficients will be the same as those in Example 12, (**377**). Hence,

$$(4a^2-3x)^{\frac{1}{4}}$$
$$=(4a^2)^{\frac{1}{4}}-\frac{(4a^2)^{-\frac{3}{4}}(3x)}{4}-\frac{3(4a^2)^{-\frac{7}{4}}(3x)^2}{4\cdot 8}-\frac{3\cdot 7(4a^2)^{-\frac{11}{4}}(3x)^3}{4\cdot 8\cdot 12}-\ldots.$$
$$=(2a)^{\frac{1}{2}}-\frac{3x}{4(2a)^{\frac{3}{2}}}-\frac{3(3x)^2}{4\cdot 8(2a)^{\frac{7}{2}}}-\frac{3\cdot 7(3x)^3}{4\cdot 8\cdot 12(2a)^{\frac{11}{2}}}-\ldots.$$
$$=\sqrt{2a}\left(1-\frac{3x}{16a^2}-\frac{27x^2}{512a^4}-\frac{567x^3}{24576a^6}-\ldots\right), \textit{ Ans.}$$

FRENCH'S THEOREM.

(**379**, page 320.)

1. Here we have $n=4$, $a=2$, $b=5$, $\dfrac{b}{a}=\dfrac{5}{2}$; hence,

$$\begin{aligned} C_1 &= (2)^4 &&= 16 \\ C_2 &= 16 \cdot \tfrac{4}{1} \cdot \tfrac{5}{2} &&= 160 \\ C_3 &= 160 \cdot \tfrac{3}{2} \cdot \tfrac{5}{2} &&= 600 \\ C_4 &= 600 \cdot \tfrac{2}{3} \cdot \tfrac{5}{2} &&= 1000 \\ C_5 &= 1000 \cdot \tfrac{1}{4} \cdot \tfrac{5}{2} &&= 625 \end{aligned}$$

Hence,

$$(2x+5y)^4=16x^4+160x^3y+600x^2y^2+1000xy^3+625y^4, \textit{ Ans.}$$

2. $n=5$, $a=2$, $b=3$, $\dfrac{b}{a}=\dfrac{3}{2}$.

$$\begin{aligned} C_1 &= (2)^5 &&= 32 \\ C_2 &= 32 \cdot \tfrac{5}{1} \cdot \tfrac{3}{2} &&= 240 \\ C_3 &= 240 \cdot \tfrac{4}{2} \cdot \tfrac{3}{2} &&= 720 \\ C_4 &= 720 \cdot \tfrac{3}{3} \cdot \tfrac{3}{2} &&= 1080 \\ C_5 &= 1080 \cdot \tfrac{2}{4} \cdot \tfrac{3}{2} &&= 810 \\ C_6 &= 810 \cdot \tfrac{1}{5} \cdot \tfrac{3}{2} &&= 243 \end{aligned}$$

Hence,

$$(2a-3x)^5=32a^5-240a^4x+720a^3x^2-1080a^2x^3+810ax^4-243x^5, \textit{ Ans.}$$

3. $n=6$, $a=3$, $b=4$, $\dfrac{b}{a}=\dfrac{4}{3}$.

$$\begin{aligned} C_1 &= (3)^6 &&= 729 \\ C_2 &= 729 \cdot \tfrac{6}{1} \cdot \tfrac{4}{3} &&= 5832 \\ C_3 &= 5832 \cdot \tfrac{5}{2} \cdot \tfrac{4}{3} &&= 19440 \\ C_4 &= 19440 \cdot \tfrac{4}{3} \cdot \tfrac{4}{3} &&= 34560 \\ C_5 &= 34560 \cdot \tfrac{3}{4} \cdot \tfrac{4}{3} &&= 34560 \\ C_6 &= 34560 \cdot \tfrac{2}{5} \cdot \tfrac{4}{3} &&= 18432 \\ C_7 &= 18432 \cdot \tfrac{1}{6} \cdot \tfrac{4}{3} &&= 4096 \end{aligned}$$

Hence,

$$(3+4x^2)^6=729+5832x^2+19440x^4+34560x^6+34560x^8 +18432x^{10}+4096x^{12}, \textit{ Ans.}$$

4. $\qquad n=4, \quad a=\frac{3}{4}, \quad b=\frac{4}{5}, \quad \frac{b}{a}=\frac{16}{15}.$

$$\begin{aligned}
C_1 &= (\tfrac{3}{4})^4 &&= \tfrac{81}{256} \\
C_2 &= \tfrac{81}{256} \cdot \tfrac{4}{1} \cdot \tfrac{16}{15} &&= \tfrac{27}{20} \\
C_3 &= \tfrac{27}{20} \cdot \tfrac{3}{2} \cdot \tfrac{16}{15} &&= \tfrac{54}{25} \\
C_4 &= \tfrac{54}{25} \cdot \tfrac{2}{3} \cdot \tfrac{16}{15} &&= \tfrac{192}{125} \\
C_5 &= \tfrac{192}{125} \cdot \tfrac{1}{4} \cdot \tfrac{16}{15} &&= \tfrac{256}{625}
\end{aligned}$$

Hence,

$$\left(\frac{3a}{4}+\frac{4r}{5}\right)^4=\frac{81}{256}a^4+\frac{27}{20}a^3r+\frac{54}{25}a^2r^2+\frac{192}{125}ar^3+\frac{256}{625}r^4, \textit{ Ans.}$$

5. $\qquad n=6, \quad a=\frac{2}{3}, \quad b=\frac{3}{2}, \quad \frac{b}{a}=\frac{9}{4}.$

$$\begin{aligned}
C_1 &= (\tfrac{2}{3})^6 &&= \tfrac{64}{729} \\
C_2 &= \tfrac{64}{729} \cdot \tfrac{6}{1} \cdot \tfrac{9}{4} &&= \tfrac{32}{27} \\
C_3 &= \tfrac{32}{27} \cdot \tfrac{5}{2} \cdot \tfrac{9}{4} &&= \tfrac{20}{3} \\
C_4 &= \tfrac{20}{3} \cdot \tfrac{4}{3} \cdot \tfrac{9}{4} &&= 20 \\
C_5 &= 20 \cdot \tfrac{3}{4} \cdot \tfrac{9}{4} &&= \tfrac{135}{4} \\
C_6 &= \tfrac{135}{4} \cdot \tfrac{2}{5} \cdot \tfrac{9}{4} &&= \tfrac{243}{8} \\
C_7 &= \tfrac{243}{8} \cdot \tfrac{1}{6} \cdot \tfrac{9}{4} &&= \tfrac{729}{64}
\end{aligned}$$

Hence,

$$\left(\frac{2t}{3}+\frac{3r}{2}\right)^6=\frac{64}{729}t^6+\frac{32}{27}t^5r+\frac{20}{3}t^4r^2+20t^3r^3+\frac{135}{4}t^2r^4+\frac{243}{8}tr^5 +\frac{729}{64}r^6, \textit{ Ans.}$$

6. $\qquad n=5, \quad a=\frac{1}{4}, \quad b=\frac{1}{5}, \quad \frac{b}{a}=\frac{4}{5}.$

$$\begin{aligned}
C_1 &= (\tfrac{1}{4})^5 &&= \tfrac{1}{1024} \\
C_2 &= \tfrac{1}{1024} \cdot \tfrac{5}{1} \cdot \tfrac{4}{5} &&= \tfrac{1}{256} \\
C_3 &= \tfrac{1}{256} \cdot \tfrac{4}{2} \cdot \tfrac{4}{5} &&= \tfrac{1}{160} \\
C_4 &= \tfrac{1}{160} \cdot \tfrac{3}{3} \cdot \tfrac{4}{5} &&= \tfrac{1}{200} \\
C_5 &= \tfrac{1}{200} \cdot \tfrac{2}{4} \cdot \tfrac{4}{5} &&= \tfrac{1}{500} \\
C_6 &= \tfrac{1}{500} \cdot \tfrac{1}{5} \cdot \tfrac{4}{5} &&= \tfrac{1}{3125}
\end{aligned}$$

Hence,

$$\left(\frac{m}{4}-\frac{1}{5}\right)^5=\frac{m^5}{1024}-\frac{m^4}{256}+\frac{m^3}{160}-\frac{m^2}{200}+\frac{m}{500}-\frac{1}{3125}, \textit{ Ans.}$$

7. $n=8, \quad a=\frac{1}{2}, \quad b=\frac{1}{2}, \quad \frac{b}{a}=1,$

$$\begin{aligned}
C_1 &= (\tfrac{1}{2})^8 &&= \tfrac{1}{256}\\
C_2 &= \tfrac{1}{256}\cdot\tfrac{8}{1}\cdot 1 &&= \tfrac{1}{32}\\
C_3 &= \tfrac{1}{32}\cdot\tfrac{7}{2} &&= \tfrac{7}{64}\\
C_4 &= \tfrac{7}{64}\cdot\tfrac{6}{3} &&= \tfrac{7}{32}\\
C_5 &= \tfrac{7}{32}\cdot\tfrac{5}{4} &&= \tfrac{35}{128}\\
C_6 &= \tfrac{35}{128}\cdot\tfrac{4}{5} &&= \tfrac{7}{32}\\
C_7 &= \tfrac{7}{32}\cdot\tfrac{3}{6} &&= \tfrac{7}{64}\\
C_8 &= \tfrac{7}{64}\cdot\tfrac{2}{7} &&= \tfrac{1}{32}\\
C_9 &= \tfrac{1}{32}\cdot\tfrac{1}{8} &&= \tfrac{1}{256}
\end{aligned}$$

Hence,

$$\left(\frac{m}{2}-\frac{1}{2m}\right)^8=\frac{m^8}{256}-\frac{m^6}{32}+\frac{7m^4}{64}-\frac{7m^2}{32}+\frac{35}{128}-\frac{7}{32m^2}+\frac{7}{64m^4}-\frac{1}{32m^6}+\frac{1}{256m^8}, \textit{ Ans.}$$

(**380**, page 323.)

1. $\sqrt[3]{9}=\sqrt[3]{8+1}=2\sqrt[3]{1+\frac{1}{8}}.$

$$\begin{aligned}
A & &&= +1.0000000\\
B &= +\tfrac{1}{3}\cdot\tfrac{1}{8}\cdot A &&= +\quad\ \ 416667\\
C &= -\tfrac{1}{3}\cdot\tfrac{1}{8}\cdot B &&= -\qquad 17361\\
D &= -\tfrac{5}{9}\cdot\tfrac{1}{8}\cdot C &&= +\qquad\ 1206\\
E &= -\tfrac{2}{3}\cdot\tfrac{1}{8}\cdot D &&= -\qquad\ \ 101\\
F &+ -\tfrac{11}{15}\cdot\tfrac{1}{8}\cdot E &&= +\qquad\quad\ 9\\
\hline
& && 1.0400420=\sqrt[3]{1+\tfrac{1}{8}}\\
& && \qquad\qquad 2\\
\hline
\sqrt[3]{9} &= && 2.080084, \textit{ Ans.}
\end{aligned}$$

2. $\sqrt[3]{31}=\sqrt[3]{27+4}=3\sqrt[3]{1+\frac{4}{27}}$.

$$
\begin{array}{rlr}
A & = & +1.0000000 \\
B = +\frac{1}{3}\cdot\frac{4}{27}\cdot A & = + & 493827 \\
C = -\frac{1}{3}\cdot\frac{4}{27}\cdot B & = - & 24387 \\
D = -\frac{5}{9}\cdot\frac{4}{27}\cdot C & = + & 2008 \\
E = -\frac{2}{3}\ \ \frac{4}{27}\cdot D & = - & 198 \\
F = -\frac{11}{15}\cdot\frac{4}{27}\cdot E & = + & 21 \\
\hline
 & & 1.0471271=\sqrt[3]{1+\frac{4}{27}} \\
 & & 3 \\
\hline
\sqrt[3]{31} & = & 3.141381,\ \textit{Ans.}
\end{array}
$$

3. $\sqrt[3]{100}=\sqrt[3]{125-25}=5\sqrt[3]{1-\frac{1}{5}}$.

$$
\begin{array}{rlr}
A & = & +1.0000000 \\
B = -\frac{1}{3}\cdot\frac{1}{5}\cdot A & = - & 666667 \\
C = -\frac{1}{3}\cdot\frac{1}{5}\cdot B & = - & 44444 \\
D = -\frac{5}{9}\cdot\frac{1}{5}\cdot C & = - & 4938 \\
E = -\frac{2}{3}\cdot\frac{1}{5}\cdot D & = - & 658 \\
F = -\frac{11}{15}\cdot\frac{1}{5}\cdot E & = - & 97 \\
G = -\frac{7}{9}\cdot\frac{1}{5}\cdot F & = - & 15 \\
H = -\frac{17}{21}\cdot\frac{1}{5}\cdot G & = - & 2 \\
\hline
 & & 0.9283179=\sqrt[3]{1-\frac{1}{5}} \\
 & & 5 \\
\hline
\sqrt[3]{100} & = & 4.641589,\ \textit{Ans.}
\end{array}
$$

4. $\sqrt[3]{110}=\sqrt[3]{125-15}=5\sqrt[3]{1-\frac{3}{25}}$.

$$
\begin{array}{rlr}
A & = & +1.0000000 \\
B = -\frac{1}{3}\cdot\frac{3}{25}\cdot A & = - & 400000 \\
C = -\frac{1}{3}\cdot\frac{3}{25}\cdot B & = - & 16000 \\
D = -\frac{5}{9}\cdot\frac{3}{25}\cdot C & = - & 1067 \\
E = -\frac{2}{3}\cdot\frac{3}{25}\cdot D & = - & 85 \\
F = -\frac{11}{15}\cdot\frac{3}{25}\cdot E & = - & 7 \\
G = -\frac{7}{9}\cdot\frac{3}{25}\cdot F & = - & 1 \\
\hline
 & & 0.9582840=\sqrt[3]{1+\frac{3}{25}} \\
 & & 5 \\
\hline
\sqrt[3]{110} & = & 4.791420,\ \textit{Ans.}
\end{array}
$$

5. $\sqrt[5]{297}=\sqrt[5]{243+54}=3\sqrt[5]{1+\frac{2}{9}}.$

$$
\begin{array}{lrr}
A & = & +1.0000000 \\
B = +\frac{1}{5}\cdot\frac{2}{9}\cdot A & = + & 444444 \\
C = -\frac{2}{5}\cdot\frac{2}{9}\cdot B & = - & 39506 \\
D = -\frac{3}{5}\cdot\frac{2}{9}\cdot C & = + & 5268 \\
E = -\frac{7}{10}\cdot\frac{2}{9}\cdot D & = - & 819 \\
F = -\frac{19}{25}\cdot\frac{2}{9}\cdot E & = + & 138 \\
G = -\frac{4}{5}\cdot\frac{2}{9}\cdot F & = - & 25 \\
H = -\frac{29}{35}\cdot\frac{2}{9}\cdot G & = + & 4 \\
\hline
 & & 1.0409504=\sqrt[5]{1+\frac{2}{9}} \\
 & & 3 \\
\hline
\sqrt[5]{297} = & & 3.122851,\ \textit{Ans.}
\end{array}
$$

6. $\sqrt[6]{60}=\sqrt[6]{64-4}=2\sqrt[6]{1-\frac{1}{16}}.$

$$
\begin{array}{lrr}
A & = & +1.0000000 \\
B = -\frac{1}{6}\cdot\frac{1}{16}\cdot A & = - & 104167 \\
C = -\frac{5}{12}\cdot\frac{1}{16}\cdot B & = - & 2713 \\
D = -\frac{11}{18}\cdot\frac{1}{16}\cdot C & = - & 103 \\
E = -\frac{17}{24}\cdot\frac{1}{16}\cdot D & = - & 5 \\
\hline
 & & 0.9893012=\sqrt[6]{1-\frac{1}{16}} \\
 & & 2 \\
\hline
\sqrt[6]{64} = & & 1.978602,\ \textit{Ans.}
\end{array}
$$

7. $\sqrt[5]{4}=\sqrt[5]{32-28}=2\sqrt[5]{1-\frac{7}{8}},$

In this case the last term, $\frac{7}{8}$, is nearly equal to unity, and the series will converge very slowly. We may give the calculation another form by taking

$$4=\frac{128}{32}, \text{ and } \sqrt[5]{4}=\frac{1}{2}\sqrt[5]{128}.$$

Whence, $\frac{1}{2}\sqrt[5]{128}=\frac{1}{2}\sqrt[5]{243-115}=\frac{3}{2}\sqrt[5]{1-\frac{115}{243}}.$

$$\begin{aligned}
A & & &= +1.0000000\\
B &= -\tfrac{1}{5}\cdot\tfrac{115}{243}\cdot A &&= - \quad 946502\\
C &= -\tfrac{4}{10}\cdot\tfrac{115}{243}\cdot B &&= - \quad 179173\\
D &= -\tfrac{9}{15}\cdot\tfrac{115}{243}\cdot C &&= - \quad 50876\\
E &= -\tfrac{14}{20}\cdot\tfrac{115}{243}\cdot D &&= - \quad 16854\\
F &= -\tfrac{19}{25}\cdot\tfrac{115}{243}\cdot E &&= - \quad 6062\\
G &= -\tfrac{24}{30}\cdot\tfrac{115}{243}\cdot F &&= - \quad 2295\\
H &= -\tfrac{29}{35}\cdot\tfrac{115}{243}\cdot G &&= - \quad 900\\
I &= -\tfrac{34}{40}\cdot\tfrac{115}{243}\cdot H &&= - \quad 362\\
J &= -\tfrac{39}{45}\cdot\tfrac{115}{243}\cdot I &&= - \quad 150\\
K &= -\tfrac{44}{50}\cdot\tfrac{115}{243}\cdot J &&= - \quad 62\\
L &= -\tfrac{49}{55}\cdot\tfrac{115}{243}\cdot K &&= - \quad 26\\
M &= -\tfrac{54}{60}\cdot\tfrac{115}{243}\cdot L &&= - \quad 11\\
N &= -\tfrac{59}{65}\cdot\tfrac{115}{243}\cdot M &&= - \quad 5\\
O &= -\tfrac{64}{70}\cdot\tfrac{115}{243}\cdot N &&= - \quad 2\\
\hline
& & & 0.8796720 = \sqrt[5]{1-\tfrac{115}{243}}\\
& & & \qquad\qquad \tfrac{3}{2}\\
\hline
& & & \sqrt[5]{4} = 1.319508, \textit{ Ans.}
\end{aligned}$$

8. $$\sqrt[5]{3275}=\sqrt[5]{3125+150}=5\sqrt[5]{1+\tfrac{6}{125}}.$$

$$\begin{aligned}
A & & &= +1.0000000\\
B &= +\tfrac{1}{5}\cdot\tfrac{6}{125}\cdot A &&= + \quad 96000\\
C &= -\tfrac{2}{5}\cdot\tfrac{6}{125}\cdot B &&= - \quad 1843\\
D &= +\tfrac{3}{5}\cdot\tfrac{6}{125}\cdot C &&= + \quad 53\\
E &= -\tfrac{7}{10}\cdot\tfrac{6}{125}\cdot D &&= - \quad 2\\
\hline
& & & 1.0094208 = \sqrt[5]{1+\tfrac{6}{125}}\\
& & & \qquad\qquad 5\\
\hline
\sqrt[5]{3275} &= & & 5.047104, \textit{ Ans.}
\end{aligned}$$

9. $$\sqrt[7]{125}=\sqrt[7]{128-3}=2\sqrt[7]{1-\tfrac{3}{128}}.$$

$$\begin{aligned}
A & & &= +1.0000000\\
B &= -\tfrac{1}{7}\cdot\tfrac{3}{128}\cdot A &&= - \quad 33482\\
C &= -\tfrac{6}{14}\cdot\tfrac{3}{128}\cdot B &&= - \quad 336\\
D &= -\tfrac{13}{21}\cdot\tfrac{3}{128}\cdot C &&= - \quad 5\\
\hline
& & & 0.9966177 = \sqrt[7]{1-\tfrac{3}{128}}\\
& & & \qquad\qquad 2\\
\hline
\sqrt[7]{125} &= & & 1.993235, \textit{ Ans.}
\end{aligned}$$

NOTE.—A table of logarithms affords an easy and practical method of finding the roots of numbers, since we have only to divide the logarithm of the number by the index of the root, and look out the number corresponding. See **404**.

To find $\sqrt[5]{4}$, we have

$$\log. 4 = 0.6020600$$

$$\frac{\log. 4}{5} = 0.1204120$$

Hence, $$\sqrt[5]{4} = 1.319508$$

as in Example 7.

METHOD OF INDETERMINATE COEFFICIENTS.

(**382**, page 327.)

1. $$\frac{1-2x}{1+3x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots.$$

$$0 = \begin{array}{r|r|r|r|r|l} A & x^0 + \;\; B & x + \;\; C & x^2 + \;\; D & x^3 + \;\; E & x^4 + \ldots. \\ -1 & -3A & -3B & -3C & -3D & \\ & +\;\;2 & & & & \end{array}$$

Therefore,

$$\begin{array}{rl} A-1=0, & A=1; \\ B-3A+2=0, & B=1; \\ C-3B=0, & C=3; \\ D-3C=0, & D=9; \\ E-3D=0, & E=27, \text{ etc.} \end{array}$$

whence,

$$1 + x + 3x^2 + 9x^3 + 27x^4 + 81x^5 + \ldots., \textit{ Ans.}$$

2. $$\frac{1+2x}{1-x-x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots.$$

$$0 = \begin{array}{r|r|r|r|r|l} A & x^0 + B & x + C & x^2 + D & x^3 + \;\; E & x^4 + \ldots. \\ -1 & -A & -A & -B & -C & \\ & -2 & -B & -C & -D & \end{array}$$

Therefore,

$$\begin{array}{rl} A-1=0, & A=1; \\ B-A-2=0, & B=3; \\ C-A-B=0, & C=4; \\ D-B-C=0, & D=7; \\ E-C-D=0, & E=11; \text{ etc.} \end{array}$$

whence,

$$1+3x+4x^2+7x^3+11x^4+18x^5+\dots, \textit{ Ans.}$$

3. $$\frac{1-x}{1-3x-2x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\dots$$

$$0=\begin{array}{r|r|r|r|r|l} A & x^0+\ B & x+\ C & x^2+\ D & x^3+\ E & x^4+\dots \\ -1 & -3A & -3B & -3C & -3D & \\ & +\ 1 & -2A & -2B & -2C & \end{array}$$

Therefore,

$$\begin{array}{rl} A-1=0, & A=1; \\ B-3A+1=0, & B=2; \\ C-3B-2A=0, & C=8; \\ D-3C-2B=0, & D=28; \\ E-3D-2C=0, & E=100; \text{ etc.} \end{array}$$

whence,

$$1+2x+8x^2+28x^3+100x^4+356x^5+\dots, \textit{ Ans.}$$

4. $$\frac{1+5x}{1-4x+4x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\dots$$

$$0=\begin{array}{r|r|r|r|r|l} A & x^0+\ B & x+\ C & x^2+\ D & x^3+\ E & x^4+\dots \\ -1 & -4A & -4B & -4C & -4D & \\ & -5 & +4A & +4B & +4C & \end{array}$$

Therefore,

$$\begin{array}{rl} A-1=0, & A=1; \\ B-4A-5=0, & B=9; \\ C-4B+4A=0, & C=32; \\ D-4C+4B=0, & D=92; \\ E-4D+4C=0, & E=240; \text{ etc.} \end{array}$$

Whence,

$$1+9x+32x^2+92x^3+240x^4+\dots$$

$$\frac{x}{x+9x^2+32x^3+92x^4+240x^5+\dots}, \textit{ Ans.}$$

5. $$\frac{2}{3x-2x^2}=Ax^{-1}+Bx^0+Cx+Dx^2+Ex^3+\ldots.$$

$$0=\begin{array}{r|r|r|r|r|l} 3A & x^0+3B & x+3C & x^2+3D & x^3+3E & x^4+\ldots. \\ -\ 2 & -2A & -2B & -2C & -2D & \end{array}$$

Therefore,

$$\begin{array}{ll} 3A-\ \ 2=0, & A=\frac{2}{3}; \\ 3B-2A=0, & B=\frac{4}{9}; \\ 3C-2B=0, & C=\frac{8}{27}; \\ 3D-2C=0, & D=\frac{16}{81}; \\ 3E-2D=0, & E=\frac{32}{243}, \text{ etc.} \end{array}$$

Whence,

$$\frac{2}{3x}+\frac{4}{9}+\frac{8x}{27}+\frac{16x^2}{81}+\frac{32x^3}{243}+\ldots., \textit{Ans.}$$

6. $$\frac{1}{1+2x^2+3x^4}=Ax^0+Bx+Cx^2+Dx^3+Ex^4+\ \ldots$$

$$0=\begin{array}{r|r|r|r|r|r|r|l} A & x^0+B & x+\ \ C & x^2+\ D & x^3+\ \ E & x^4+\ \ F & x^5+\ \ G & x^6+\ldots. \\ -\ 1 & & +2A & +2B & +2C & +2D & +2E & \\ & & & & +3A & +3B & +3C & \end{array}$$

Therefore,

$$\begin{array}{lll} A-\ \ 1=0, & A=\ \ \ 1; & \\ B\ \ \ \ =0, & B=\ \ \ 0; & \\ C+2A=0, & C=-\ 2; & \\ D+2B=0, & D=\ \ \ 0; & \\ 3A+E+2C=0, & E=\ \ \ 1; & \\ 3B+F+2D=0, & F=\ \ \ 0; & \\ 3C+G+2E=0, & G=\ \ \ 4; & (7) \\ 3E+I+2G=0, & I=-11; & \\ 3G+K+2I=0, & K=\ \ 10; & \\ 3I+M+2K=0 & M=\ \ 13, \text{ etc.} & \end{array}$$

Whence,

$$1-2x^2+x^4+4x^6-11x^8+10x^{10}+13x^{12}-\ldots., \textit{Ans.}$$

NOTE.—It is obvious that that the alternate equations may be omitted; this is done from the seventh onward.

7. $$\frac{1+x}{1+2ax+a^2x^2}=Ax^0+Bx+Cx^2+Dx^3+Ex^4+\ldots.$$

$$0=\begin{array}{r|c|c|c|c|c|l} A & x^0+\quad B & x+\quad C & x^2+\quad D & x^3+\quad E & x^4+\quad F & x^5+\ldots. \\ -1 & +2aA & +2aB & +2aC & +2aD & +2aE & \\ & -\quad 1 & +a^2A & +a^2B & +a^2C & +a^2D & \end{array}$$

Therefore,

$$\begin{aligned} A-1&=0, & A&=1; \\ B+2aA-1&=0, & B&=+(1-2a); \\ C+2aB+a^2A&=0, & C&=-(2a-3a^2); \\ D+2aC+a^2B&=0, & D&=+(3a^2-4a^3); \\ E+2aD+a^2C&=0, & E&=-(4a^3-5a^4); \text{ etc.} \end{aligned}$$

Whence,

$$1+(1-2a)x-(2a-3a^2)x^2+(3a^2-4a^3)x^3-(4a^3-5a^4)x^4+\ldots,$$ *Ans.*

8. $$\sqrt{1-x}=A+Bx+Cx^2+Dx^3+Ex^4+\ldots.$$

Squaring both sides, we have

$$1-x=\begin{array}{r|c|c|c|l} A^2+AB & x+AC & x^2+AD & x^3+AE & x^4+\ldots. \\ +AB & +B^2 & +BC & +BD & \\ & +AC & +BC & +C^2 & \\ & & +AD & +BD & \\ & & & +AE & \end{array}$$

Therefore,

$$\begin{aligned} A^2-1&=0, & A&=\ \ 1; \\ 2AB+1&=0, & B&=-\tfrac{1}{2}; \\ 2AC+B^2&=0, & C&=-\frac{1}{2\cdot4}; \\ 2AD+2BC&=0, & D&=-\frac{3}{2\cdot4\cdot6}; \\ 2AE+C^2+2BD&=0, & E&=-\frac{3\cdot5}{2\cdot4\cdot6\cdot8}; \text{ etc.} \end{aligned}$$

Hence,

$$\sqrt{1-x}=1-\frac{x}{2}-\frac{x^2}{2\cdot4}-\frac{3x^3}{2\cdot4\cdot6}-\frac{3\cdot5x^4}{2\cdot4\cdot6\cdot8}-\ldots,$$ *Ans.*

9. Assuming the same form of development as in Example 8, we shall have,

$$
\begin{aligned}
A^2-1 &= 0, & A &= 1; \\
2AB-3 &= 0, & B &= \tfrac{3}{2}; \\
2AC+B^2-5 &= 0, & C &= \tfrac{11}{8}; \\
2AD+2BC-7 &= 0, & D &= \tfrac{23}{16}; \\
2AE+C^2+2BD-9 &= 0, & E &= \tfrac{179}{128}; \text{ etc.}
\end{aligned}
$$

Whence, $$1+\frac{3x}{2}+\frac{11x^2}{8}+\frac{23x^3}{16}+\frac{179x^4}{128}+\ldots., \textit{Ans.}$$

10.

$$\frac{1-2x^2+3x^4-4x^6+5x^8-6x^{10}+\ldots.}{1+x^2+x^4+x^6+x^8+x^{10}+\ldots.}=Ax^0+Bx^2+Cx^4+Dx^6+\ldots.$$

$$
\begin{array}{r|c|c|c|c|c|c|l}
0= \quad A & x^0+B & x^2+C & x^4+D & x^6+E & x^8+F & x^{10}+\ldots. \\
-1 & +A & +B & +C & +D & +E & \\
 & +2 & +A & +B & +C & +D & \\
 & & -3 & +A & +B & +C & \\
 & & & +4 & +A & +B & \\
 & & & & -5 & +A & \\
 & & & & & +6 &
\end{array}
$$

$$
\begin{aligned}
A-1 &= 0, & A &= 1; \\
B+A+2 &= 0, & B &= -3; \\
C+B+A-3 &= 0, & C &= +5; \\
D+C+B+A+4 &= 0, & D &= -7; \text{ etc.}
\end{aligned}
$$

Whence, $$1-3x^2+5x^4-7x^6+9x^8-11x^{10}+\ldots., \textit{Ans.}$$

REVERSION OF SERIES.

(383, page 330.)

1. Here we have $a=1$, $b=1$, $c=1$, etc.; whence by formula (F),

$$A=1, \quad B=-1 \quad C=1, \quad D=-1, \quad E=1, \text{ etc.}$$

Hence,

$$x=y-y^2+y^3-y^4+y^5-\ldots.; \textit{ Ans.}$$

2. $a=1, \quad b=3, \quad c=5,$ etc.

By formula (T),

$$A=1, \quad B=-3, \quad C=13, \quad D=-67, \quad E=381, \text{ etc.}$$

Whence,

$$x=y-3y^2+13y^3-67y^4+381y^5-\ldots., \text{ } Ans.$$

3. $a=1, \quad b=-\frac{1}{2}, \quad c=\frac{1}{3}, \quad d=-\frac{1}{4}, \quad e=\frac{1}{5},$ etc.

By formula (F),

$$A=1, \quad B=\frac{1}{2}, \quad C=\frac{1}{2\cdot 3}, \quad D=\frac{1}{2\cdot 3\cdot 4}, \quad E=\frac{1}{2\cdot 3\cdot 4\cdot 5}, \text{ etc.}$$

Whence,

$$y=x+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}+\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\ldots., \text{ } Ans.$$

4. $a=1, \quad b=-1, \quad c=1, \quad d=-1, \quad e=1,$ etc.

By formula (G),

$$A=1, \quad B=1, \quad C=2, \quad D=5, \quad E=14, \text{ etc.}$$

whence,

$$x=y+y^3+2y^5+5y^7+14y^9+\ldots., \text{ } Ans.$$

5. $a=2, \quad b=3, \quad c=4, \quad d=5, \quad e=6,$ etc.

By formula (G),

$$A=\tfrac{1}{2}, \quad B=-\tfrac{3}{16}, \quad C=\tfrac{19}{128}, \quad D=-\tfrac{152}{1024}, \text{ etc.}$$

Whence,

$$x=\frac{y}{2}-\frac{3y^3}{16}+\frac{19y^5}{128}-\frac{152y^7}{1024}+\ldots., \text{ } Ans.$$

6. $a=2, \quad b=4, \quad c=6, \quad d=8, \quad e=10,$ etc.

By formula (F),

$$A=\tfrac{1}{2}, \quad B=-\tfrac{1}{2}, \quad C=\tfrac{5}{8}, \quad D=-\tfrac{7}{8}, \quad E=\tfrac{21}{16}, \text{ etc.}$$

whence,

$$y=\frac{x}{2}-\frac{x^2}{2}+\frac{5x^3}{8}-\frac{7x^4}{8}+\frac{21x^5}{16}-\ldots.$$

(**384**, page 331.)

In the following examples, let s be the sum of the series.

1. By formula (F),

$$x=\frac{1}{5}s+\frac{4}{25}s^2+\frac{2\cdot 8}{125}s^3+\frac{4\cdot 16}{625}s^4+\frac{8\cdot 32}{3125}s^5+\ldots.$$

Substituting the value of $s=\frac{2}{5}$,

$$x=\tfrac{1}{2}(\tfrac{2}{5})^2+(\tfrac{2}{5})^4+2(\tfrac{2}{5})^6+4\cdot(\tfrac{2}{5})^8+8\cdot(\tfrac{2}{5})^{10}+\ldots.$$

This is a geometrical series in which $r=2(\frac{2}{5})^2$; and by **361**,

$$x=\frac{\frac{1}{2}(\frac{2}{5})^2}{1-2(\frac{2}{5})^2}=0.11764706+,\ \textit{Ans.}$$

2. By formula (F), we have

$$x=s-\tfrac{1}{6}s^2-\tfrac{1}{36}s^3-\tfrac{1}{270}s^4-\tfrac{1}{3240}s^5-\ldots..$$

Substituting the value of $s=\frac{1}{2}$, we find

$$\begin{aligned} s &= +0.500000\\ -\tfrac{1}{6}s^2 &= -0.041667\\ -\tfrac{1}{36}s^3 &= -0.003472\\ -\tfrac{1}{270}s^4 &= -0.000231\\ -\tfrac{1}{3240}s^5 &= -0.000010\end{aligned}$$

Therefore, $x=+0.454620$, *Ans.*

3. By formula (G), we have

$$x=s+\tfrac{1}{6}s^3+\tfrac{13}{120}s^5+\tfrac{493}{5040}s^7+\ldots..$$

Hence,

$$\begin{aligned} s &= +0.200000\\ +\tfrac{1}{6}s^3 &= +0.001333\\ +\tfrac{13}{120}s^5 &= +0.000035\\ \tfrac{493}{5040}s^7 &= +\qquad\quad 1\end{aligned}$$

Therefore, $x=\ 0.201369$, *Ans.*

4. By formula (F),

$$x=s+\tfrac{3}{8}s^2+\tfrac{7}{96}s^3+\tfrac{87}{4608}s^4+\tfrac{1229}{307200}s^5+\ldots\ldots$$

Hence,

$$\begin{aligned} s &= +0.250000 \\ \tfrac{3}{8}s^2 &= +0.023438 \\ \tfrac{7}{96}s^3 &= +0.001139 \\ \tfrac{87}{4608}s^4 &= +0.000074 \\ \tfrac{1229}{307200}s^5 &= +0.000004 \\ \hline \end{aligned}$$

Therefore, $x= \quad .274655$, *Ans.*

RECURRING SERIES.

(**391**, page 335.)

1. We have by formula (P),

$$m+3n=4,$$
$$3m+4n=7;$$

whence,

$$m=1,$$
$$n=1.$$

By formula (Q),

$$S=\frac{1+3x-x}{1-x-x^2}=\frac{1+2x}{1-x-x^2}, \textit{ Ans.}$$

2. By formula (P), we have

$$m+6n=12,$$
$$6m+12n=48;$$

whence,

$$m=6,$$
$$n=1.$$

By formula (Q),

$$S=\frac{1+6x-x}{1-x-6x^2}=\frac{1+5x}{1-x-6x^2}, \textit{ Ans.}$$

3. By formula (P), we have

$$m+2n=-\ 5,$$
$$2m-5n=+26;$$

whence,

$$m=+\ 3,$$
$$n=-\ 4.$$

Hence by (Q),

$$S=\frac{1+2x+4x}{1+4x-3x^2}=\frac{1+6x}{1+4x-3x^2}, \textit{ Ans.}$$

4. By formula (T), we have

$$m+4n+3r=-\ 2,$$
$$4m+3n-2r=+\ 4,$$
$$3m-2n+4r=+17;$$

whence,

$$m=+\ 3,$$
$$n=-\ 2,$$
$$r=+\ 1.$$

Hence by (V),

$$S=\frac{1+4x+3x^2-(1+4x)x+2x^2}{1-x+2x^2-3x^3}=\frac{1+3x+x^2}{1-x+2x^2-3x^3}, \textit{ Ans.}$$

5. By formula (P), we have

$$m+3n=\ \ 5,$$
$$3m+5n=\ \ 7;$$

whence,

$$m=-1,$$
$$n=+2.$$

By (Q),

$$S=\frac{1+3x-2x}{1-2x+x^2}=\frac{1+x}{(1-x)^2}, \textit{ Ans.}$$

6. By formula (P), we have

$$m+n=\ 5,$$
$$m+5n=13;$$

whence,

$$m=\ 3,$$
$$n=\ 2.$$

By (Q),

$$S=\frac{1+x-2x}{1-2x-3x^2}=\frac{1-x}{1-2x-3x^2}, \textit{ Ans.}$$

7. By formula (T), we have

$$\begin{aligned} m+4n+6r &= 11, \\ 4m+6n+11r &= 28, \\ 6m+11n+28r &= 63; \end{aligned}$$

whence,

$$\begin{aligned} m &= +3, \\ n &= -1, \\ r &= +2. \end{aligned}$$

By (V),

$$S=\frac{1+4x+6x^2-(1+4x)\cdot 2x+x^2}{1-2x+x^2-3x^3}$$

$$=\frac{1+2x-2x^2+x^3}{1-2x+x^2-3x^3}=\frac{(1+x)^2-2x^3}{(1-x)^2-3x^3}, \text{ Ans.}$$

8. By formula (P), we have

$$\begin{aligned} \frac{m}{2}+n &= \frac{7}{2}, \\ m+\frac{7n}{2} &= 10; \end{aligned}$$

whence,

$$\begin{aligned} m &= 3, \\ n &= 2. \end{aligned}$$

By (Q) we may now sum the series, but we must notice that equations (2), (**389**), will now have the form,

$$\begin{aligned} C &= mAx^4+nBx^2, \\ D &= mBx^4+nCx^2, \text{ etc.}; \end{aligned}$$

or that we must change x in (Q) to x^2. Hence,

$$S=\frac{\frac{x}{2}+x^3-x^3}{1-2x^2-3x^4}=\frac{x}{2-4x^2-6x^4}, \text{ Ans.}$$

DIFFERENTIAL METHOD.

(**395**, page 339.)

1. $a=1, \quad d_1=3, \quad d_2=1, \quad d_3=0.$

By formula (A),

$$T_9=1+3\cdot8+\frac{8\cdot7}{2}=53, \textit{ Ans.}$$

2. $a=1, \quad d_1=3, \quad d_2=3, \quad d_3=1, \quad d_4=0.$

By formula (A),

$$T_{15}=1+3\cdot14+\frac{14\cdot13}{2}\cdot3+\frac{14\cdot13\cdot12}{2\cdot3}=680, \textit{ Ans.}$$

3. $a=1, \quad d_1=5, \quad d_2=10, \quad d_3=10, \quad d_4=5, \quad d_5=0.$

$$T_8=1+5\cdot7+\frac{7\cdot6}{2}\cdot10+\frac{7\cdot6\cdot5}{2\cdot3}\cdot10+\frac{7\cdot6\cdot5\cdot4}{2\cdot3\cdot4}\cdot5$$

$$T_9=1+5\cdot8+\frac{8\cdot7}{2}\cdot10+\frac{8\cdot7\cdot6}{2\cdot3}\cdot10+\frac{8\cdot7\cdot6\cdot5}{2\cdot3\cdot4}\cdot5$$

Hence, $T_8=781$, and $T_9=1231$, *Ans.*

4. $a=1, \quad d_1=7, \quad d_2=12, \quad d_3=6, \quad d_4=0.$

$$T_{20}=1+7\cdot19+\frac{19\cdot18}{2}\cdot12+\frac{19\cdot18\cdot17}{2\cdot3}\cdot6=8000, \textit{ Ans.}$$

5. $a=1, \quad d_1=2, \quad d_2=1, \quad d_3=0.$

$$T_n=1+2(n-1)+\frac{(n-1)(n-2)}{2}$$

$$=\frac{2+4n-4+n^2-3n+2}{2}=\frac{n(n+1)}{2}, \textit{ Ans.}$$

6. $a=1, \quad d_1=3, \quad d_2=3, \quad d_3=1, \quad d_4=0.$

$$T_n=1+3(n-1)+\frac{(n-1)(n-2)}{2}\cdot 3+\frac{(n-1)(n-2)(n-3)}{2\cdot 3}$$

$$=\frac{9n^2-9n+6+n^3-6n^2+11n-6}{6}$$

$$=\frac{n^3+3n^2+2n}{6}=\frac{n(n+1)(n+2)}{6}, \textit{ Ans.}$$

7. $a=1, \quad d_1=4, \quad d_2=6, \quad d_3=4, \quad d_4=1.$

$$T_n=1+4(n-1)+\frac{(n-1)(n-2)}{2}\cdot 6+\frac{(n-1)(n-2)(n-3)}{2\cdot 3}\cdot 4+$$

$$\frac{(n-1)(n-2)(n-3)(n-4)}{2\cdot 3\cdot 4}.$$

Or, $$T_n=\frac{n^4+6n^3+11n^2+6n}{24}=\frac{n(n+3)(n^2+3n+2)}{24}$$

$$=\frac{n(n+3)(n+2)(n+1)}{24}, \textit{ Ans.}$$

8. $a=1, \quad d_1=2, \quad d_2=1, \quad d_3=0.$

By formula (B), (**394**),

$$S=20+\frac{20\cdot 19}{2}\cdot 2+\frac{20\cdot 19\cdot 18}{2\cdot 3}=1540, \textit{ Ans.}$$

9 $a=1, \quad d_1=4, \quad d_2=5, \quad d_3=2, \quad d_4=0.$

$$S=12+\frac{12\cdot 11}{2}\cdot 4+\frac{12\cdot 11\cdot 10}{2\cdot 3}\cdot 5+\frac{12\cdot 11\cdot 10\cdot 9}{2\cdot 3\cdot 4}\cdot 2;$$

or $$S=2366, \textit{ Ans.}$$

10. $a=1, \quad d_1=3, \quad d_2=6, \quad d_3=9, \quad d_4=0.$

$$S=10+\frac{10\cdot 9}{2}\cdot 3+\frac{10\cdot 9\cdot 8}{2\cdot 3}\cdot 6+\frac{10\cdot 9\cdot 8\cdot 7}{2\cdot 3\cdot 4}\cdot 9;$$

or, $$S=2755, \textit{ Ans.}$$

11. $a=2, \quad d_1=4, \quad d_2=2, \quad d_3=0.$

$$S=2n+\frac{n(n-1)}{2}\cdot 4+\frac{n(n-1)(n-2)}{2\cdot 3}\cdot 2; \text{ or}$$

$$S=\frac{n^3+3n^2+2}{3}=\frac{n(n+2)(n+1)}{3}, \textit{ Ans.}$$

12. $a=6, \quad d_1=18, \quad d_2=18, \quad d_3=6.$

$$S=6n+\frac{n(n-1)}{2}\cdot 18+\frac{n(n-1)(n-2)}{2\cdot 3}\cdot 18+\frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}\cdot 6;$$

$$S=\frac{n^4+6n^3+11n^2+6n}{4}=\frac{n(n+3)(n+2)(n+1)}{4}, \textit{ Ans.}$$

13. The series in examples 13, 14, and 15 may be summed by formula (B), (**389**); but the following method by indeterminate coefficients is easier.

Assume

$$1^2+2^2+3^2+\ldots\ldots+n^2=A+Bn+Cn^2+Dn^3+En^4+\ldots\ldots \quad (1)$$

Change n into $n+1$; then

$$\begin{aligned}1^2+2^2+3^2+\ldots\ldots+n^2+(n+1)^2=A+B(n+1)+C(n+1)^2&\\ +D(n+1)^3+E(n+1)^4+\ldots\ldots& \quad (2)\end{aligned}$$

By subtraction,

$$\begin{aligned}n^2+2n+1=B+C(2n+1)+D(3n^2+3n+1)&\\ +E(4n^3+6n^2+4n+1)+\ldots\ldots&\end{aligned}$$

or, by arranging terms with reference to the powers of n,

$$\begin{aligned}n^2+2n+1=4En^3+(3D+6E)n^2+(2C+3D+4E)n&\\ +(B+C+D+E)+\ldots\ldots& \quad (3)\end{aligned}$$

Now by (**368**, III.), we have

$$4E=0, \text{ or } E=0;$$

and the same value for all coefficients beyond E.

Equating coefficients of like powers of n in (3), omitting the terms containing E, we have the following equations of condition:

$$3D=1, \quad \text{whence} \quad D=\tfrac{1}{3};$$
$$2C+3D=2, \quad \text{"} \quad C=\tfrac{1}{2};$$
$$B+C+D=1, \quad \text{"} \quad B=\tfrac{1}{6}.$$

And by substitution in (1),

$$1^2+2^2+3^2+\ldots\ldots+n^2=A+\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3},$$

To determine A, put $n=1$; then $A=0$, and we have

$$S=\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}=\frac{2n^3+3n^2+n}{6}=\frac{n(n+1)(2n+1)}{6}, \textit{ Ans.}$$

In Example 14, we shall have $F=0$, and the same for every coefficient beyond F. The equations of condition are

$$4E=1, \quad \text{whence} \quad E=\tfrac{1}{4};$$
$$3D+6E=3, \quad \text{"} \quad D=\tfrac{1}{2};$$
$$2C+3D+4E=3, \quad \text{"} \quad C=\tfrac{1}{4};$$
$$B+C+D+E=1, \quad \text{"} \quad B=0.$$

Hence,

$$S=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}=\frac{(n^2+n)^2}{4}, \textit{ Ans.}$$

In Example 15, we have $G=0$, and the same for every coefficient beyond G. The equations of condition are

$$5F=1, \quad \text{whence} \quad F=\tfrac{1}{5},$$
$$4E+10F=4, \quad \text{"} \quad E=\tfrac{1}{2},$$
$$3D+6E+10F=6, \quad \text{"} \quad D=\tfrac{1}{3},$$
$$2C+3D+4E+\ 5F=4, \quad \text{"} \quad C=0,$$
$$B+C+D+E+F=1, \quad \text{"} \quad B=-\tfrac{1}{30}.$$

Hence, $$S=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}, \textit{ Ans.}$$

In this solution the symmetry of the equations that determine F, E, D, &c., should be noticed. Taking the numbers in vertical columns, they are the binomial coefficients. We can easily extend this method to the fifth and sixth powers; and so on.

16. Performing the multiplications indicated, we shall have the two series,

$$m(1+2+3+\ldots\ldots+n)$$

and, $$1^2+2^2+3^2+\ldots\ldots+n^2.$$

Hence, $$S=\frac{n(n+1)}{2}m+\frac{n(n+1)(2n+1)}{6};$$

or, $$S=\frac{n(n+1)(1+2n+3m)}{6}, \textit{ Ans.}$$

INTERPOLATION.

(**397**, page 341.)

For the first, second, and third examples, we have

$a=2.758924$, $d_1=+.043115$, $d_2=-.001287$, $d_3=+.000091$.

For the fourth, fifth, and sixth,

$a=2.802039$, $d_1=+.041828$, $d_2=.001196$, $d_3=+.000083$.

By substitution in the formula, we obtain the following results:

1.

1st term,	+	2.758924
2d "	+	14012
3d "	+	141
4th "	+	6
		2.773083, *Ans.*

2.

1st term,	+	2.758924
2d "	+	37726
3d "	+	70
4th "	+	2
		2.796722, *Ans.*

3.

1st term,	+	2.758924
2d "	+	19695
3d "	+	160
4th "	+	6
		2.778785, *Ans.*

4.

1st term,	+	2.802039
2d "	+	10457
3d "	+	112
4th "	+	5
		2.812613, *Ans.*

5.

1st term,	+	2.802039
2d "	+	28611
3d "	+	129
4th "	+	4
		2.820783, *Ans.*

6.

1st term,	+	2.802039
2d "	+	31371
3d "	+	112
4th "	+	3
		2.833525, *Ans.*

(**398**, page 342.)

FUNCTION.	d_1	d_2	d_3
66° 6′ 38″			
72 24 5	+6° 17′ 27″		
78 34 48	+6 10 43	−6′ 44″	
84 39 4	+6 4 16	−6 27	+17″
90 37 18	+5 58 14	−6 2	+25
96 29 57	+5 52 39	−5 35	+27

NOTE.—In forming the differences we may use as a check to guard against numerical errors, the rule, "The sum of the differences in any column, plus the first term of the preceding column, is equal to the last term of the preceding column."

Observe that for the 1st, 2d, and 3d examples,

$$a = 66^\circ\, 6'\, 38'',\quad d_1 = +6^\circ\, 17'\, 27'',\quad d_2 = -6'\, 44'',\quad d_3 = +17''.$$

For the 4th, 5th, and 6th,

$$a = 72^\circ\, 24'\, 5'',\quad d_1 = +6^\circ\, 10'\, 43'',\quad d_2 - 6'\, 27'',\quad d_3 = +25''.$$

For the 7th, 8th, and 9th,

$$a = 78^\circ\, 34'\, 48'',\quad d_1 = +6^\circ\, 4'\, 16'',\quad d_2 = -6'\, 2'',\quad d_3 = +27''.$$

Again, we have

for examples 1st, 4th, and 7th, $n = \frac{1}{4}$;
" " 2d, 5th, and 8th, $n = \frac{1}{2}$;
" " 3d, 6th, and 9th, $n = \frac{3}{4}$.

1. 1st term, +66° 6′ 38″
2d " + 1 34 22
3d " + 38
4th " + 1
67° 41′ 39″, *Ans.*

2. 1st term, +66° 6′ 38″
2d " + 3 8 43.5
3d " + 50.5
4th " + 1
69° 16′ 13″, *Ans.*

3. 1st term, $+66° \ 6' \ 38''$
2d " $+ \ 4 \ 43 \ 5$
3d " $+ \ 38$
4th " $+ \ 1$

$70° \ 50' \ 22''$, *Ans.*

4. 1st term, $+72° \ 24' \ 5''$
2d " $+ \ 1 \ 32 \ 41$
3d " $+ \ 36$
4th " $+ \ 1$

$73° \ 57' \ 23''$, *Ans.*

5. 1st term, $+72° \ 24' \ 5''$
2d " $+ \ 3 \ 5 \ 21.5$
3d " $+ \ 48.4$
4th " $+ \ 1$

$75° \ 30' \ 16''$, *Ans.*

6. 1st term, $+72° \ 24' \ 5''$
2d " $+ \ 4 \ 38 \ 2$
3d " $+ \ 36$
4th " $+ \ 1$

$77° \ 2' \ 44''$, *Ans.*

7. 1st term, $+78° \ 34' \ 48''$
2d " $+ \ 1 \ 31 \ 4$
3d " $+ \ 34$
4th " $+ \ 1$

$80° \ 6' \ 27''$, *Ans.*

8. 1st term, $+78° \ 34' \ 48''$
2d " $+ \ 3 \ 2 \ 8$
3d " $+ \ 45$
4th " $+ \ 2$

$81° \ 37' \ 43''$, *Ans.*

9. 1st term, $+78° \ 34' \ 48''$
2d " $+ \ 4 \ 33 \ 12$
3d " $+ \ 34$
4th " $+ \ 1$

$83° \ 8' \ 35''$, *Ans.*

LOGARITHMS.

(**416**, page 355.)

2. log. 104=2.017033
" 5=0.698970

2.716003, *Ans.*

3. log. 73=1.863323
" 2=0.301030

2.164353, *Ans.*

4. log. 50 = 1.698970
" 29 = 1.462398
3.161368, *Ans.*

5. log. .53 = −1.724276
" 3 = 0.477121
0.201397, *Ans.*

6. log. 1017 = 3.007321
" 2 = 0.301030
3.308351, *Ans.*

7. log. 10.91 = 1.037825
" 7 = 0.845098
1.882923

8. log. .01005 = −2.002166
" 2 = 0.301030
2.303196, *Ans.*

9. log. .91 = −1.959041
" .42 = −1.623249
−1.582290, *Ans.*

10. log. 103 = 2.012837
" 15 = 1.176091
" 11 = 1.041393
4.230321

(**417**, page 356.)

3. log. 10720 = 4.030195
405 × 0.4 = 162
4.030357, *Ans.*

4. log. 10.85 = 1.035430
400 × 0.39 = 156
1.035586, *Ans.*

5. log. 1021 = 3.009026
425 × 0.56 = 238
3.009264, *Ans.*

6. log. 5.6 = 0.748188
log. 101.5232 = 2.006565
2.754753, *Ans.*

7.

$$
\begin{array}{rl}
\log. 3 = & 0.477121 \\
\log. 1081.333 = & 3.033960 \\
\hline
& 3.511081, \textit{ Ans.}
\end{array}
$$

8.

$$
\begin{array}{rl}
\log. 3.6 = & 0.556303 \\
\log. 101.4601 = & 2.006295 \\
\hline
& 2.562598, \textit{ Ans.}
\end{array}
$$

9.

$$
\begin{array}{rl}
\log. 1.3 = & 0.113943 \\
\log. 101.977 = & 2.008502 \\
\hline
& 2.122445, \textit{ Ans.}
\end{array}
$$

10.

$$
\begin{array}{rl}
\log. 5.6 = & 0.748188 \\
\log. 101343.04 = & 5.005794 \\
\hline
& 5.754982, \textit{ Ans.}
\end{array}
$$

11.

$$
\begin{array}{rl}
\log. 2.5 = & 0.397940 \\
\log. 103.48 = & 2.014856 \\
\hline
& 2.412796, \textit{ Ans.}
\end{array}
$$

12.

$$
\begin{array}{rl}
\log. 1.2 = & 0.079181 \\
\log. 1.08 = & 0.033424 \\
\hline
& 0.112605, \textit{ Ans.}
\end{array}
$$

13.

$$
\begin{array}{rl}
\log. 57 = & 1.755875 \\
\log. 101.4737 = & 2.006353 \\
\hline
& 3.762228, \textit{ Ans.}
\end{array}
$$

EXPONENTIAL EQUATIONS.

(**418**, page 357.)

1. Taking the logarithms,

$$x \log. 7 = \log. 8$$

$$x = \frac{\log. 8}{\log. 7} = \frac{0.903090}{0.845098} = 1.06862, \textit{ Ans.}$$

2. Taking the logarithms,

$$\frac{2}{x}(\log. 5) = \log. 30.$$

$$x = \frac{2 \log. 5}{\log. 30} = \frac{1.397940}{1.477121} = 0.946395, \textit{ Ans.}$$

3. $$x \log. a = 2 \log. b + 3 \log. c$$

whence, $$x = \frac{2 \log. b + 3 \log. c}{\log. a}, \textit{ Ans.}$$

4. Clearing of fractions and transposing, we have

$$ab^x = dm + c$$

$$\log. a + x \log. b = \log. (dm + c)$$

whence, $$x = \frac{\log. (dm + c) - \log. a}{\log. b}, \textit{ Ans.}$$

5. $$\log. m + \frac{1}{x}(\log. a) = \log. b,$$

whence, $$x = \frac{\log. a}{\log. b - \log. m}, \textit{ Ans.}$$

6. Adding and subtracting the equations,

$$a^x = c + d, \qquad \text{and } b^y = c - d;$$

whence, $$x = \frac{\log. (c + d)}{\log. a}, \qquad y = \frac{\log. (c - d)}{\log. b}, \textit{ Ans.}$$

7. Taking the logarithms,

$$x=\frac{\log. 729}{\log. 3}=\frac{\log. 3^6}{\log. 3}=\frac{6 \log. 3}{\log. 3}=6, \textit{ Ans.}$$

8. $$\frac{3}{x}(\log. 216)=\log. 12,$$

whence, $$x=\frac{3 \log. 216}{\log. 12}=\frac{3 \log. 6^3}{\log. 12}=\frac{9 \log. 6}{\log. 12}, \textit{ Ans.}$$

9. $$\frac{3}{x}(\log. 516)=\log. 12;$$

whence, $$x=\frac{3 \log. 516}{\log. 12}=\frac{3 \log. 43+3 \log. 12}{\log. 12}$$

$$=\frac{3 \log. 43}{\log. 12}+3, \textit{ Ans.}$$

10. Taking the logarithms,

$$x \log. 6=6 \log. 24+\tfrac{1}{3} \log. 17-\log. 71$$

whence, $$x=\frac{18 \log. 24+\log. 17-3 \log. 71}{3 \log. 6}, \textit{ Ans.}$$

PROPERTIES OF EQUATIONS.

(**428**, page 366.)

1. Factors, $\begin{cases} x-2=0 \\ x+3=0 \end{cases}$

Product, $x^2-x-0=0$, *Ans.*

2. By (**423**), the 1st term is x^3;
" (**424**, 1) " 2d " $5x^2$;
" (**424**, 2) " 3d " $2x$;
" (**424**, 5) " 4th " -8.

Hence, $x^3+5x^2+2x-8=0$, *Ans.*

3. Factors, $\begin{cases} x-3=0 \\ x+2=0 \\ x+1=0 \\ x+5=0 \end{cases}$

Product, $x^4-5x^3-7x^2+29x+30=0$, *Ans.*

4. Factors,

$$\begin{cases} x-\left(1+\sqrt{-5}\right)=0 & (1) \\ x-\left(1-\sqrt{-5}\right)=0 & (2) \\ x-\sqrt{5} \qquad =0 & (3) \\ x+\sqrt{5} \qquad =0 & (4) \end{cases}$$

Product of (1) and (2), $x^2-2x \qquad =6$ (5)

" (3) " (4), $x^2-5 \qquad =0$ (6)

" (5) " (6), $x^4-2x^3+x^2+10x-30=0$, *Ans.*

5. By (**423**,) The 1st term is x^5;

" (**424**, 1), " 2d " $-4x^4$;

" (**424**, 2), " 3d " $0x^3$;

" (**424**, 3), " 4th " $22x^2$;

" (**424**, 4), " 5th " $-26x$;

" (**424**, 5), " 6th " -42.

Hence, the required equation is

$$x^5-4x^4+22x^2-25x-42=0, \textit{ Ans.}$$

Or thus:

Factors,

$$\begin{cases} x+1=0 & (1) \\ x+2=0 & (2) \\ x-3=0 & (3) \\ x-\left(2+\sqrt{-3}\right)=0 & (4) \\ x-\left(2-\sqrt{-3}\right)=0 & (5) \end{cases}$$

Product of (4) and (5), $x^2-4x+7=0$.

Hence,

$$(x^2-4x+7)(x+1)(x+2)(x-3)=x^5-4x^4\pm0x^3+22x^2-25x-42=0,$$

Ans.

6. $(x^3-5x^2+13x-21)\div(x-3)=x^2-2x+7=0$, *Ans.*

7. $(x^4+2x^3-34x^2+12x+35)\div(x+7)=x^3-5x^2+x+5=0$, *Ans.*

8. $$(x-2)(x+3)=x^2+x-6\text{; hence,}$$

$$(x^4-3x^3-4x^2+30x-36)\div(x^2+x-6)=x^2-4x+6=0,$$

the depressed equation, which may be solved thus:

$$x^2-4x=-6,$$
$$x^2-4x+4=-2,$$
$$x-2=\pm\sqrt{-2},$$
$$x=2+\sqrt{-2}, \text{ or } 2-\sqrt{-2}.$$

COMMENSURABLE ROOTS.

(432, page 373.)

2.

Divisors,	6,	3,	2,	1,	−1,	−2,	−3,	−6.
	−6							
Quotients,	−1,	−2,	−3,	−6,	6,	3,	2,	1.
Add	11							
	10,	9,	8,	5,	17,	14,	13,	12.
2d quotients,		3,	4,	5,	−17,	−7,		−2.
Add		−6						
		−3,	−2,	−1,	−23,	−13,		−8.
3d quotients,		−1,	−1,	−1,	23,			

There are three final quotients equal to -1; and the corresponding divisors are 3, 2, and 1. Hence the given equation has three commensurable roots, 1, 2, and 3, *Ans.*

3. Divisors,	12,	6,	4,	3,	2,	1,	− 1,	− 2,	− 3,	− 4,	−6,	−12.
	−12											
Quotients,	− 1,	− 2,	− 3,	− 4,	− 6,	−12,	12,	6,	4,	3,	2,	1.
Add,	−16											
	−17,	−18,	−19,	−20,	−22,	−28,	− 4,	−10,	−12,	−13,	−14,	−15.
2d quotients,		− 3,			−11,	−28,	4,	5,	4.			
Add,		− 1,										
		− 4,			−12,	−29,	3,	4,	3.			
3d quotients,					− 6,	−29,	− 3,	− 2,	− 1.			
Add,					+ 4							
					− 2,	−25,	1,	2,	3.			
4th quotients,					− 1,	−25,	− 1,	− 1,	− 1.			

There are four final quotients equal to −1; and the corresponding divisors are 2, −1, −2, and −3. Hence, the equation has four commensurable roots, 2, −1, −2, and −3; and as the equation is only of the fourth degree, these are all the roots.

4.

Divisors,	21,	7,	3,	1,	−1,	−3,	−7,	−21.
	21							
Quotients,	1,	3,	7,	21,	−21,	−7,	−3,	−1.
Add	−16							
	−15,	−13,	−9,	5,	−37,	−23,	−19,	−17.
2d quotients,			−3,	5,	37.			
Add			−6					
			−9,	−1,	31.			
3d quotients,			−3,	−1,	−31.			
Add			±0					
			−3,	−1,	−31.			
4th quotients,			−1,	−1,	31.			

The final quotients show that the given equation has two commensurable roots, 3, and 1, *Ans.*

5.

Divisors,	10,	5,	2,	1,	−1,	−2,	−5,	−10.
	−10							
Quotients,	−1,	−2,	−5,	−10,	10,	5,	2,	1.
Add	2							
	1,	0,	−3,	−8,	12,	7,	4,	3.
2d quotients,		0,		−8,	−12.			
Add		5						
		5,		−3,	−7.			
3d quotients,		1,		−3,	7.			
Add		−6						
		−5,		−9,	1.			
4th quotients,		−1,		−9,	−1.			

The final quotients show that 5 and -1 are the commensurable roots of the given equation. Hence,

$$x-5=0, \qquad (1)$$
$$x+1=0. \qquad (2)$$

Dividing the given equation by the product of (1) and (2), we have

$$\begin{aligned}(x^4-6x^3+5x^2+2x-10)\div(x^2-4x-5)=x^2-2x+2&=0,\\ x^2-2x+1&=-1,\\ x-1&=\pm\sqrt{-1},\\ x&=1\pm\sqrt{-1}.\end{aligned}$$

Therefore the four roots are 5, -1, $1+\sqrt{-1}$, and $1-\sqrt{-1}$, *Ans.*

EQUAL ROOTS.

(**435**, page 379.)

2. We have given

$$X=x^5+2x^4-11x^3-8x^2+20x+16=0.$$

The first derived polynomial of the first member is

$$X_1=5x^4+8x^3-33x^2-16x+20.$$

The greatest common divisor of this, and the first member of the given equation found by (**105**), is

$$D=x^2-x-2=(x-2)(x+1).$$

Therefore $x=2$ is twice a root of the equation, also $x=-1$ is twice a root; and the equation has two roots, each equal to 2; and two roots, each equal to -1.

Dividing $x^5+2x^4-11x^3-8x^2+20x+16=0$, by $(x-2)^2(x+1)^2=0$, we have $x+4=0$, and $x=-4$. Hence the roots of the given equation are, 2, 2, -1, -1, -4.

3. We have given

$$X=x^5-2x^4+3x^3-7x^2+8x-3=0;$$

whence,

$$X_1=5x^4-8x^3+9x^2-14x+8=0.$$

By (**105**) we obtain

$$D=x^2-2x+1=(x-1)^2=0.$$

Therefore, since $x=1$ is twice a root of D, by (**435**, II) it is *three times* a root of the given equation X.

4. $$X = x^4-2x^3-11x^2+12x+36=0;$$
$$X_1=4x^3-6x^2-22x+12=0.$$

Whence, by (**105**),

$$D=x^2-x-6=(x-3)(x+2)=0.$$

Hence $x=3$ is twice a root of X, and also $x=-2$; and as the equation is of the fourth degree, it has only four roots. Therefore,

3, 3, -2, -2, *Ans.*

TRANSFORMATIONS.

(**437**, page 384.)

5. Put $x=y+2$; then $x'=2$. Whence,

$$X' = (2)^4-4(2)^3-8(2)+32, \quad \text{or} \quad X' = 0;$$

$$X'_1= 4(2)^3-12(2)^2-8, \quad \text{or} \quad X'_1=-24;$$

$$\frac{X'_2}{2}=\frac{3\cdot4(2)^2}{2}-\frac{2\cdot12(2)}{2}, \quad \text{or} \quad \frac{X'_2}{2}= 0;$$

$$\frac{X'_3}{2\cdot3}=\frac{2\cdot3\cdot4(2)}{2\cdot3}-\frac{2\cdot12}{2\cdot3}, \quad \text{or} \quad \frac{X'_3}{2\cdot3}= 4;$$

$$\frac{X'_4}{2\cdot3\cdot4}=\frac{2\cdot3\cdot4}{2\cdot3\cdot4}, \quad \text{or} \quad \frac{X'_4}{2\cdot3\cdot4}= 1.$$

Therefore the transformed equation must be

$$y^4+4y^3-23y=0, \text{ *Ans.*}$$

6. Put $x=y-3$; then $x'=-3$. Whence

$$X' = (-3)^4+16(-3)^3+99(-3)^2+228(-3)+144, \quad \text{or} \quad X' = 0;$$

$$X'_1 = 4(-3)^3+48(-3)^2+198(-3)+228, \quad \text{or} \quad X'_1 = -42;$$

$$\frac{X'_2}{2} = \frac{3\cdot4(-3)^2}{2}+\frac{2.48(-3)}{2}+\frac{198}{2}, \quad \text{or} \quad \frac{X'_2}{2} = 9;$$

$$\frac{X'_3}{2\cdot3} = \frac{2\cdot3\cdot4(-3)}{2\cdot3}+\frac{2\cdot48}{2\cdot3}, \quad \text{or} \quad \frac{X'_3}{2.3} = 4;$$

$$\frac{X'_4}{2\cdot3\cdot4} = \frac{2\cdot3\cdot4}{2\cdot3\cdot4}, \quad \text{or} \quad \frac{X'_4}{2.3.4} = 1.$$

Therefore, the transformed equation must be

$$y^4+4y^3+9y^2-42y=0, \ \textit{Ans.}$$

7. By (**437**), put $x=y+\frac{3}{4}$, then $x'=2$. Whence,

$$X' = (2)^4-8(2)^3+(2)^2+82(2)-60 = 60;$$

$$X'_1 = 4(2)^3-24(2)^2+2(2)+82 = 22;$$

$$\frac{X'_2}{2} = \frac{3\cdot4(2)^2}{2}-\frac{2\cdot24(2)}{2}+\frac{2}{2} = -23;$$

$$\frac{X'_3}{2\cdot3} = \frac{2\cdot3\cdot4(2)}{2\cdot3}-\frac{2\cdot24}{2\cdot3} = 0;$$

$$\frac{X'_4}{2.3.4} = \frac{2\cdot3\cdot4}{2\cdot3\cdot4} = 1.$$

Therefore, the transformed equation must be

$$y^4-23y^2+22y+60=0, \ \textit{Ans.}$$

MULTIPLICATION AND DIVISION BY DETACHED COEFFICIENTS.

(**440**, page 390.)

4.

$$\begin{array}{r} 3-2-1 \\ 4+2 \\ \hline 12-8-4 \\ +6-4-2 \\ \hline 12-2-8-2 \end{array}$$

5.

$$\begin{array}{r} 3-5-10 \\ 2-4 \\ \hline 6-10-20 \\ -12+20+40 \\ \hline 6-22\pm\ 0+40 \end{array}$$

6.
$$\begin{array}{l}
1+1+1 \\
\underline{1-1+1} \\
1+1+1 \\
\quad -1-1-1 \\
\qquad\quad +1+1+1 \\
\hline
1\pm0+1\pm0+1
\end{array}$$

7.
$$\begin{array}{l}
1-4+5-2 \\
\underline{1+4-3} \\
1-4+\ 5-\ 2 \\
\quad +4-16+20-8 \\
\qquad\quad -\ 3+12-15+\ 6 \\
\hline
1\pm0-14+30-23+16
\end{array}$$

SYNTHETIC DIVISION.

(**442**, page 394.)

1.
$$\begin{array}{l}
1-1 \ \underline{|-1\qquad} \\
\quad -1+2-2+2 \\
\hline
1-2+2-2+2, \text{ etc.}
\end{array}$$

Hence the quotient is $1-2x+2x^2-2x^3+2x^4-$, &c., *Ans.*

2.
$$\begin{array}{l}
1 \ \underline{|-1\qquad\qquad} \\
\quad -1+1-1+1 \\
\hline
1-1+1-1+1, \text{ \&c.}
\end{array}$$

Hence the quotient is $1-x+x^2-x^3+x^4-$, &c., *Ans.*

3.
$$\begin{array}{l}
1-5+10-10+5-1 \ \underline{|2-1} \\
\quad +2-\ 6+\ 6-2 \\
\qquad\quad -\ 1+\ 3-2+1 \\
\hline
1-3+\ 3-\ 1\ \ 0\ \ 0
\end{array}$$

Hence the quotient is $a^3-3a^2x+3ax^2-x^3$, *Ans.*

4.
$$\begin{array}{l}
1-5+15-24+27-13+5 \ \underline{|2-4+2-1} \\
\quad +2-\ 6+10 \\
\qquad\quad -\ 4+12-20 \\
\qquad\qquad\quad +\ 2-\ 6+10 \\
\qquad\qquad\qquad\quad -\ 1+\ 3-5 \\
\hline
1-3+\ 5\quad 0\quad 0\quad 0\quad 0.
\end{array}$$

Hence the quotient is x^2-3x+5, *Ans.*

5.
$$
\begin{array}{l}
1 \pm 0 \pm 0 \pm 0 \pm 0 \pm 0 \pm 0 - 1 \,\underline{|1} \\
\quad 1 + 1 + 1 + 1 + 1 + 1 + 1 \\
\hline
1 + 1 + 1 + 1 + 1 + 1 + 1 + 0.
\end{array}
$$

Hence the quotient is $x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6$, *Ans.*

CARDAN'S RULE FOR CUBIC EQUATIONS.

(**449**, page 404.)

4. We have
$$3p = -6, \quad \text{or} \quad p = -2;$$
$$2q = 6, \quad \text{or} \quad q = 3;$$
$$\sqrt{q^2 + p^3} = \sqrt{9 - 8} = -1.$$

Whence by formula (A),
$$x = (3+1)^{\frac{1}{3}} + (3-1)^{\frac{1}{3}} = \sqrt[3]{4} + \sqrt[3]{2} = 2.8473+, \text{ Ans.}$$

5. Here,
$$3p = 9, \quad \text{or} \quad p = 3;$$
$$2q = 6, \quad \text{or} \quad q = 3;$$
$$\sqrt{q^2 + p^3} = \sqrt{9 + 27} = \sqrt{36} = \pm 6,$$

Whence by formula (A),
$$x = (3+6)^{\frac{1}{3}} + (3-6)^{\frac{1}{3}} = \sqrt[3]{9} + \sqrt[3]{-3} = .63783+, \text{ Ans.}$$

6. To transform this equation into another deficient of its second term, according to (**437**), put $x = y - 2$, and we shall have for the transformed equation,
$$y^3 - 25y + 66 = 0.$$

Hence, in applying Cardan's rule,
$$3p = -25, \quad \text{or} \quad p = -\tfrac{25}{3};$$
$$2q = -66, \quad \text{or} \quad q = -33;$$
$$\sqrt{q^2 + p^3} = \sqrt{1089 - \tfrac{15625}{27}} = \pm 22.589737+.$$

Whence by formula (A),

$$y=\left(-33+22.589737\right)^{\frac{1}{3}}+\left(-33-22.589737\right)^{\frac{1}{3}}$$
$$=\ -2.18-3.82=-6.$$

Therefore, $$x=-6-2=-8.$$

Dividing the given equation by $(x+8)$, we have for the depressed equation,

$$x^2-2x+3=0;$$

whence, $$x=1\pm\sqrt{-2}.$$

Hence, the three roots are

$$-8,\ 1+\sqrt{-2},\ \text{and}\ 1-\sqrt{-2},\ \textit{Ans.}$$

LIMITS OF REAL ROOTS.

(**453**, page 408.)

2. Here, $n=2$, and $P=25$; hence,

$$\sqrt[n]{P}+1=\sqrt{25}+1=6,\ \textit{Ans.}$$

3. Supplying the deficient term, we have

$$x^4+0x^3-5x^2-9x+10=0.$$

Therefore, $n=2$, and $P=9$; hence,

$$\sqrt[n]{P}+1=\sqrt{9}+1=4,\ \textit{Ans.}$$

4. Here $n=3$, and $P=8$; hence,

$$\sqrt[n]{P}+1=\sqrt[3]{8}+1=3,\ \textit{Ans.}$$

(**454**, page 408).

1. Changing the signs of the alternate terms, we have

$$x^3+3x^2+5x-7=0.$$

Therefore, $n=3$, and $P=7$; and

$$\sqrt[n]{P}+1=\sqrt[3]{7}+1=3,\ \text{in whole numbers, } \textit{Ans.}$$

2. Completing the terms, we have

$$x^4 \pm 0x^3 - 15x^2 - 10x + 24 = 0.$$

Changing the signs of the alternate terms, the equation becomes

$$x^4 \mp 0x^3 - 15x^2 + 10x + 24 = 0.$$

The term having zero for a coefficient may always be regarded as positive. Hence

$$n = 2, \quad \text{and } P = 15.$$

$$\sqrt[2]{P} + 1 = \sqrt{15} + 1 = 5, \; Ans.$$

3. With the alternate signs changed, the equation is

$$x^6 + 3x^5 + 2x^4 - 27x^3 - 4x^2 + 1 = 0.$$

Hence, $\quad n = 3, \quad$ and $P = 27.$

$$\sqrt[3]{P} + 1 = \sqrt[3]{27} + 1 = 4, \; Ans.$$

HORNER'S METHOD OF APPROXIMATION.

(**464**, page 420.)

3. We have given

$$X = x^3 + 2x^2 - 23x - 70.$$

The first derived polynomial is

$$X_1 = 3x^2 + 4x - 23.$$

We multiply X by 3, and divide the result by X_1. Thus,

$$\begin{array}{l|l}
3x^3 + 6x^2 - 69x - 210 & 3x^2 + 4x - 23 \\
\cline{2-2}
\underline{3x^3 + 4x^2 - 23x} & x, \; +1 \\
\cline{2-2}
\quad 2x^2 - 46x - 210 & \\
\quad\; x^2 - 23x - 105 & \\
\quad 3x^2 - 59x - 315, \text{ new prepared dividend.} & \\
\quad \underline{3x^2 + \;\, 4x - \;\, 23} & \\
\qquad -73x - 292. \quad \text{Hence, } R = x + 4. &
\end{array}$$

We now divide X_1 by R. Thus,

$$\begin{array}{r|l} 3x^2+\ 4x-23 & x+4 \\ \cline{2-2} \underline{3x^2+12x\quad\ \ } & \underline{3x-8} \\ -\ 8x-23 & \\ \underline{-\ 8x-32} & \\ +\ 9. & \end{array} \quad \text{Hence, } R_1=-9.$$

Therefore, the functions are as follows:

$$\begin{aligned} X\ &=x^3+2x^2-23x-70, \\ X_1&=3x^2+4x-23, \\ R\ &=x+4, \\ R_1&=-9. \end{aligned}$$

Substituting in these functions, $x=-\infty$ and $x=+\infty$ successively, we have the following results, in respect to signs:

	X	X_1	R	R_1	
For $x=-\infty$,	$-$	$+$	$-$	$-$,	2 variations.
For $x=+\infty$,	$+$	$+$	$+$	$-$,	1 variation.

Hence the given equation has 1 real root.

If we substitute $x=0$ in the functions, the signs will be

$$-\quad -\quad +\quad -,$$

giving 2 variations. Hence the real root must lie between 0 and $+\infty$; or, it is positive. To ascertain its position, make $x=1$, $x=2$, $x=3$, etc., successively.

For $x=1$, signs,	$-$	$-$	$+$	$-$,	2 variations.
$x=2$, "	$-$	$-$	$+$	$-$,	2 "
$x=3$, "	$-$	$+$	$+$	$-$,	2 "
$x=4$, "	$-$	$+$	$+$	$-$,	2 "
$x=5$, "	$-$	$+$	$+$	$-$,	2 "
$x=6$, "	$+$	$+$	$+$	$-$,	1 variation,

Hence, the initial figure is 5. The decimal part is found by the following operation:

(420, Ex. 3)

1	+2	−23	−70	5.1345787253
	5	35	60	
	7	12	(1) −10.000	
	5	60	7.371	
	12	(1) 72.00	(2) −2.629000	
	5	1.71	2.278497	
	(1) 17.0	73.71	(3) −.350503000	
	.1	1.72	.306161104	
	17.1	(2) 75.4300	(4) −44341896	
	.1	.5199	38309285	
	17.2	75.9499	(5) −6032611	
	.1	.5208	5363994	
	(2) 17.30	(3) 76.470700	(6) −668617	
	3	69576	613038	
	17.33	72.540276	(7) −55579	
	3	69592	53641	
	17.36	(4) 76.609868	(8) −1938	
	3	8701	1533	
	(3) 17.390	76.618569	(9) −405	
	4	8701	383	
	17.394	(5) 76.62727	(10) −22	
	4	122	23	
	17.398	76.62849		
	4	122		
	(4) 17.402	(6) 76.6297		
	(5) 17.4	1		
	(6) 1	76.6298		
		1		
		(7) 76.630		
		(8) 76.63		
		(9) 76.6		
		(10) 77		

(420, Ex. 3)

4. We have given

$$X=x^3-x^2+70x-300.$$

The first derived polynomial is

$$X_1=3x^2-2x+70.$$

Multiplying X by 3 to avoid fractions in dividing, we proceed as follows:

$$\begin{array}{rrrr|l}
3x^3-3x^2+210x- & 900 & & & 3x^2-2x+70 \\
3x^3-2x^2+\ \ 70x & & & & x,\ -1 \\ \hline
-\ x^2+140x- & 900 & & & \\
-3x^2+420x- & 2700, & \text{prepared dividend.} & & \\
-3x^2+\ \ \ 2x- & 70 & & & \\ \hline
+418x- & 2630. & \text{Hence, } R=-209x+1315. & &
\end{array}$$

To avoid fractions in the next division, multiply X_1 by 209.

$$\begin{array}{rr|l}
627x^2-\ \ 418x+ & 14630 & -209x+1315 \\
627x^2-3945x & & -3x,\quad -3527 \\ \hline
+3527x+ & 14630 & \\
\text{Multiplying by 209,}\quad +737143x- & 3057670 & \\
+737143x- & 4038005 & \\ \hline
+ & 1580335. & R_1=-1580335.
\end{array}$$

Hence, the functions are

$$\begin{aligned}
X\ &=\ \ x^3-x^2+70x-300,\\
X_1&=\ \ 3x^2-2x+70,\\
R\ &=-209x+1315,\\
R_1&=-1580335.
\end{aligned}$$

Let $x=-\infty$; we have	$-$	$+$	$+$	$-$,	2	variations;
" $x=+\infty$, "	$+$	$+$	$-$	$-$,	1	variation;
" $x=\ \ 0$, "	$-$	$+$	$+$	$-$,	2	variations;

Hence, the given equation has but one real root, and this lies between 0 and $+\infty$; it is therefore positive.

Let $x=1$; we have	$-$	$+$	$+$	$-$,	2	variations,
" $x=2$ "	$-$	$+$	$+$	$-$,	2	"
" $x=3$ "	$-$	$+$	$+$	$-$,	2	"
" $x=4$ "	$+$	$+$	$+$	$-$,	1	".

(420, Ex. 4)

Hence, the initial figure is 3. The decimal part is found by the following operation:

1	−1	+70	−300	3.7387936878
	3	6	228	
	2	76	(1) −72.000	
	3	15	67.963	
	5	(1) 91.00	(2) −4.037000	
	3	6.09	3.119217	
	(1) 8.0	97.09	(3) −.917783000	
	.7	6.58	.834882272	
	8.7	(2) 103.6700	(4) −82900728	
	.7	.3039	73114357	
	9.4	103.9739	(5) −9786371	
	7	3048	9401144	
	(2) 10.10	(3) 104.278700	(6) −385227	
	3	81584	313374	
	10.13	104.360284	(7) −71853	
	3	81648	62675	
	10.16	(4) 104.441932	(8) −9178	
	3	7150	8357	
	(3) 10.190	104.449082	(9) −831	
	8	7150	732	
	10.198	(5) 104.45623	(10) −89	
	8	92	84	
	10.206	104.45715	−5	
	8	92		
	(4) 10.214	(6) 104.4581		
	(5) 10.2	(7) 104.458		
	(6) 1	(8) 104.46		
		(9) 104.5		
		(10) 105		

(420, Ex. 4)

5. We have given

$$X=x^3+x^2-500.$$

The first derived polynomial is

$$X_1=3x^2+2x.$$

Multiply X by 3 to avoid fractions in dividing.

$$\begin{array}{l|l} 3x^3+3x^2-1500 & 3x^2+2x \\ \underline{3x^3+2x^2} & x,\ +1 \\ \quad x^2-1500 & \\ \quad 3x^2-4500 & \\ \quad \underline{3x^2+2x} & \\ \quad -2x-4500 & \end{array}$$

Hence, $R=x+2250$

$$\begin{array}{l|l} 3x^2+\quad 2x & x+2250 \\ \underline{3x^2+6750x} & 3x-6748 \\ \quad -6748x & \\ \quad \underline{-6748x-15183000} & \\ \qquad +15183000. & \end{array}$$

Hence, $R_1=-15183000.$

Thus the functions are

$$\begin{aligned} X &= x^3+x^2-500, \\ X_1 &= 3x^2+2x, \\ R &= x+2250, \\ R_1 &= -15183000. \end{aligned}$$

Let $x=-\infty$;	we have	$-$	$+$	$-$	$-$,	2 variations.
" $x=+\infty$;	"	$+$	$+$	$+$	$-$,	1 variation.
" $x=\quad 0$;	"	$-$	$\pm$	$+$	$-$,	2 variations.

Hence, there is but one real root; and this is positive, because it lies between 0 and $+\infty$.

Let $x=1$;	signs,	$-$	$+$	$+$	$-$,	2 variations.
" $x=2$;	"	$-$	$+$	$+$	$-$,	2 "
" $x=3$;	"	$-$	$+$	$+$	$-$,	2 "
" $x=4$	"	$-$	$+$	$+$	$-$,	2 "
" $x=5$;	"	$-$	$+$	$+$	$-$,	2 "
" $x=6$;	"	$-$	$+$	$+$	$-$,	2 "
" $x=7$;	"	$-$	$+$	$+$	$-$,	2 "
" $x=8$;	"	$+$	$+$	$+$	$-$,	1 variation.

(420, Ex. 5)

Hence, the initial figure is 7; the decimal part is found as follows:

1	+1	0	−500 |7.6172797559
	7	56	392
	8	56	(1) −108.000
	7	105	104.736
	15	(1) 161.00	(2) −3.264000
	7	13.56	1.887181
	(1) 22.0	174.56	(3) −1.376819000
	.6	13.92	1.323862113
	22.6	(2) 188.4800	(4) −.052956887
	.6	.2381	37858967
	23.2	188.7181	(5) −15097920
	.6	.2382	13251090
	(2) 23.80	(3) 188.956300	(6) −1846830
	.01	.166859	1703728
	23.81	189.123159	(7) −143102
	.01	.166908	132512
	23.82	(4) 189.290067	(8) −10590
	.01	4770	9465
	(3) 23.830	189.294837	(9) −1125
	.007	4770	947
	23.837	(5) 189.29961	(10) −178
	.007	167	170
	23.844	189.30128	−8
	.007	167	
	(4) 23.851	(6) 189.3029	
		2	
	(5) 23.9	189.3031	
		2	
	(6) 2	(7) 189.303	
		(8) 189.30	
		(9) 189.3	
		(10) 189	

(420, Ex. 5)

6. We have given,

$$X = x^3 - x^2 - 40x - 108.$$

The first derived polynomial is

$$X_1 = 3x^2 - 2x - 40.$$

Multiply X by 3, to avoid fractions in dividing.

$$\begin{array}{rl}
3x^3 - 3x^2 - 120x + 324 & \big| \; 3x^2 - 2x - 40 \\
\underline{3x^3 - 2x^2 - 40x } & \big| \; \underline{x, \; -1} \\
- x^2 - 80x + 324 & \\
\text{Multiply by 3,} \quad -3x^2 - 240x + 972 & \\
\underline{-3x^2 + 2x + 40} & \\
-242x + 932. & \text{Hence, } R = 121x - 466.
\end{array}$$

Multiplying X_1 by 121, to avoid fractions in the next division,

$$\begin{array}{rl}
363x^2 - 242x - 4840 & \big| \; 121x - 466 \\
\underline{366x^2 - 1398x} \phantom{- 4840} & \big| \; \underline{3x, \; +289} \\
+ 1156x - 4840 & \\
\text{Dividing by 4,} \quad + 289x - 1210 & \\
\text{Multiplying by 121,} \quad + 34969x - 146410 & \\
\underline{+ 34969x - 134674} & \\
- 11736 & = -R_1
\end{array}$$

Thus we find

$$\begin{aligned}
X &= x^3 - x^2 - 40x + 108, \\
X_1 &= 3x^2 - 2x - 40, \\
R &= 121x - 466, \\
R_1 &= +11736.
\end{aligned}$$

Let $x = -\infty$; we have	$-$	$+$	$-$	$+$,	3	variations
" $x = +\infty$; "	$+$	$+$	$+$	$+$,	0	"
" $x = 0$; "	$+$	$-$	$-$	$+$,	2	"

Hence, there are three real roots, two between 0 and $+\infty$, and and one between 0 and $-\infty$; that is, there are two positive roots, and one negative root.

To find the situation of the positive roots, let

$x=1$; signs,	$+$	$-$	$-$	$+$,	2	variations.
$x=2$; "	$+$	$-$	$-$	$+$,	2	"
$x=3$; "	$+$	$-$	$-$	$+$,	2	"
$x=4$; "	$-$	$\pm$	$+$	$+$,	1	variation.
$x=5$; "	$+$	$+$	$+$	$+$,	0	"

Hence the initial figures of the positive roots are 3 and 4.

To find the situation of the negative root, let

$x=-2$; signs,	$+$	$-$	$-$	$+$,	2	variations.
$x=-4$; "	$+$	$+$	$-$	$+$,	2	"
$x=-6$; "	$+$	$+$	$-$	$+$,	2	"
$x=-8$; "	$-$	$+$	$-$	$+$,	3	"

Hence the negative root is situated between 6 and 8.

Let $x=-6$; signs,	$+$	$+$	$-$	$+$,	2	variations.
" $x=-7$; "	$-$	$+$	$-$	$+$,	3	"

Hence, the initial figure of the negative root is -6,

The decimal part of the first root is obtained thus:

1	−1	−40	+108	3.3792053825
	3	6	−102	
	2	−34	(1) 6.000	
	3	15	−4.953	
	5	(1) −19.00	(2) 1.047000	
	3	2.49	−.931147	
	(1) 8.0	−16.51	(3) .115853000	
	.3	2.58	−.113285061	
	8.3	(2) −13.9300	(4) 2567939	
	.3	.6279	−2500650	
	8.6	−13.3021	(5) 67289	
	3	.6328	−62507	
	(2) 8.90	(3) −12.669300	(6) 4782	
	7	82071	−3750	
	8.97	−12.587229	(7) 1032	
	7	82152	−1000	
	9.04	(4) −12.505077	(8) 32	
	7	1827	−25	
	(3) 9.110	−12.503250	(9) 7	
	9	1827	−7	
	9.119	(5) −12.50142	0	
	9	5		
	9.128	−12.50137		
	9	5		
	(4) 9.137	(6) −12.501		
	(5) 9.1	(7) −12.50		
		(8) −12.5		
		(9) −13.		

(420, Ex. 6)

The decimal part of the second root is obtained thus:

1	−1		−40		+108	4.5875359541
	4		12		−112	
	3		−28	(1)	−4.000	
	4		28		2.875	
	7	(1)	0.00	(2)	−1.125000	
	4		5.75		1.020512	
(1)	11.0		5.75	(3)	−.104488000	
	.5		6.00		97009003	
	11.5	(2)	11.7500	(4)	−7478997	
	.5		1.0064		6977044	
	12.0		12.7564	(5)	−501953	
	.5		1.0128		418826	
(2)	12.50	(3)	13.769200	(6)	−83127	
	8		89229		69807	
	12.58		13.858429	(7)	−13320	
	8		89278		12565	
	12.66	(4)	13.947707	(8)	−755	
	8		6381		698	
(3)	12.740		13.954088	(9)	−57	
	7		6381		56	
	12.747	(5)	13.96047	(10)	−1	
	7		38		1	
	12.754		13.96085		0	
	7		38			
(4)	12.761	(6)	13.9612			
			1			
(5)	12.8		13.9613	(8)	13.96	
			1	(9)	14.0	
(6)	1	(7)	13.961	(10)	14.	

NOTE.—To obtain the trial figure for the second place in the root, proceed according to NOTE 5, (**464**). Thus, we shall have

$$y=\sqrt{-\frac{U'}{S'}}=\sqrt{\frac{4.00}{11.0}}=\sqrt{.36}=.6$$

By trial, we find that this must be diminished by .1, making .5 for the second figure of the root.

(**420**, Ex. 6)

To obtain the negative root, we may change the signs of the alternate terms; the result obtained will then be *positive*, according to Note 3, (**464**). This is not necessary, however; for we may derive the negative root directly from the given coefficients, as will be seen from the following operation:

1	−1	−40	+108 | −6.9667413367
	−6	+42	−12
	−7	2	(1) 96.000
	−6	78	−88.119
	−13	(1) 80.00	(2) 7.881000
	− 6	17.91	−7.076136
	(1) −19.0	97.91	(3) .804864000
	− .9	18.72	−.716256696
	−19.9	(2) 116.6300	(4) 88607304
	− .9	1.3056	−83665958
	−20.8	117.9356	(5) 4941346
	− 9	1.3092	−4781560
	(2) −21.70	(3) 119.244800	(6) 159786
	− 6	.131316	−119540
	−21.76	119.376116	(7) 40246
	− 6	.131352	−35862
	−21.82	(4) 119.507468	(8) 4384
	− 6	15329	−3586
	(3) −21.880	119.522797	(9) 798
	− 6	15329	−717
	−21.886	(5) 119.53812	(10) 81
	− 6	87	−83
	−21.892	119.53899	8
	− 6	87	
	(4) −21.898	(6) 119.5399	
	(5) −21.9	(7) 119.540	
	(6) −2	(8) 119.54	
		(9) 119.5	
		(10) 120	

(**420**, Ex. 6)

7. We have given

$$X = x^3 - 4x^2 - 24x + 48.$$

The first derived polynomial is

$$X_1 = 3x^2 - 8x - 24.$$

Multiplying X by 3 to avoid fractions in dividing,

$$\begin{array}{r|l}
3x^3 - 12x^2 - 72x + 144 & 3x^2 - 8x - 24 \\ \cline{2-2}
\underline{3x^3 - 8x^2 - 24x} & x,\ -1 \\
-\ 4x^2 - 48x + 144 & \\
\text{Multiplying by } \tfrac{3}{4}, \quad -\ 3x^2 - 36x + 108 & \\
\underline{-\ 3x^2 + 8x + 24} & \\
-44x + 84. & \text{Hence, } R = 11x - 21.
\end{array}$$

Multiplying X_1 by 11, to avoid fractions,

$$\begin{array}{r|l}
33x^2 - 88x - 264 & 11x - 21 \\ \cline{2-2}
\underline{33x^2 - 63x} & 3x,\ -25 \\
-\ 25x - 264 & \\
\text{Multiplying by 11}, \quad -275x - 2904 & \\
\underline{-275x + 525} & \\
-3429. & \text{Hence, } R_1 = 3429.
\end{array}$$

Hence, we have the functions,

$$\begin{aligned}
X &= x^3 - 4x^2 - 24x + 48, \\
X_1 &= 3x^2 - 8x - 24, \\
R &= 11x - 21, \\
R_1 &= 3429.
\end{aligned}$$

Let $x = -\infty$;	we have	$-$	$+$	$-$	$+$,	3	variations
" $x = +\infty$;	"	$+$	$+$	$+$	$+$,	0	"
" $x = 0$;	"	$+$	$-$	$-$	$+$,	2	"

Hence, there are three real roots, — two situated between 0 and $+\infty$ and therefore positive, and one situated between 0 and $-\infty$, and therefore negative.

Let $x = 1$;	we have	$+$	$-$	$-$	$+$,	2	variations,
" $x = 2$;	"	$-$	$-$	$+$	$+$,	1	"
" $x = 3$;	"	$-$	$-$	$+$	$+$,	1	"
" $x = 4$;	"	$-$	$-$	$+$	$+$,	1	"
" $x = 5$;	"	$-$	$+$	$+$	$+$,	1	"
" $x = 6$;	"	$-$	$+$	$+$	$+$,	1	"
" $x = 7$;	"	$+$	$+$	$+$	$+$,	0	"
" $x = -1$;	"	$+$	$-$	$-$	$+$,	2	"

(420, Ex. 7)

Let $x=-2$;	we have	$+$	$+$	$-$	$+$,	2 variations.
" $x=-3$;	"	$+$	$+$	$-$	$+$,	2 "
" $x=-4$;	"	$+$	$+$	$-$	$+$,	2 "
" $x=-5$;	"	$-$	$+$	$-$	$+$,	3 "

By inspecting the column of variations, we perceive that one root is between 1 and 2, another between 6 and 7, and another between -4 and -5. Hence, the initial figures of the three roots are

$$1, \quad 6, \quad -4.$$

The first root is obtained as follows:

1	−4	−24	+48 ⎸1.7191292611
	1	− 3	−27
	−3	−27	(1) 21.000
	1	− 2	−20.447
	−2	(1) −29.00	(2) .553000
	1	− .21	−.289189
	(1) −1.0	−29.21	(3) .263811000
	.7	+ .28	−.260077041
	− .3	(2) −28.9300	(4) 3733959
	.7	+ 111	−2888700
	+ .4	−28.9189	(5) 845259
	.7	+ 112	−577737
	(2) 1.10	(3) −28.907700	(6) 267522
	1	+ 10251	−259981
	1.11	−28.897449	(7) 7541
	1	+ 10332	−5777
	1.12	(4) −28.887117	(8) 1764
	1	+ 116	−1733
	(3) 1.130	−28.887001	(9) 31
	9	+ 116	−29
	1.139	(5) −28.88688	(10) 2
	9	+ 2	−3
	1.148	−28.88686	
	9	+ 2	(8) −28.89
	(4) 1.157	(6) −28.8868	(9) −28.9
	(5) 1.2	(7) −28.887	(10) −29.

(420, Ex. 7)

The decimal part of the second root is found thus:

1	−4	−24	+48	6.5461457261
	6	12	−72	
	2	12	(1) −24.000	
	6	48	21.625	
	8	(1) 36.00	(2) −2.375000	
	6	7.25	2.054864	
	(1) 14.0	43.25	(3) −.320136000	
	.5	7.50	.312531336	
	14.5	(2) 50.7500	(4) −7604664	
	.5	.6216	5218391	
	15.0	51.3716	(5) −2386273	
	.5	.6232	2087444	
	(2) 15.50	(3) 51.994800	(6) −298829	
	4	93756	260934	
	15.54	52.088556	(7) −37895	
	4	93792	36531	
	15.58	(4) 52.182348	(8) −1364	
	4	1564	1044	
	(3) 15.620	52.183912	(9) −320	
	6	1564	313	
	15.626	(5) 52.18548	(10) −7	
	6	62	5	
	15.632	52.18610	−2	
	6	62		
	(4) 15.638	(6) 52.1867		
		1		
	(5) 15.6	52.1868		
		1		
	(6) 2	(7) 52.187		
		(8) 52.19		
		(9) 52.2		
		(10) 52		

(420, Ex. 7)

And the decimal part of the third root is found, without changing the signs of the alternate terms of the equation, as follows:

1	−4	−24	+48	−4.2652749871
	−4	+32	−32	
	−8	8	(1) 16.000	
	−4	48	−11.848	
	−12	(1) 56.00	(2) 4.152000	
	− 4	3.24	−3.811176	
	(1) −16.0	59.24	(3) .340824000	
	− .2	3.28	−.323033625	
	−16.2	(2) 62.5200	(4) 17790375	
	− .2	9996	−12938807	
	−16.4	63.5196	(5) 4851568	
	− .2	1.0032	−4528900	
	(2) −16.60	(3) 64.522800	(6) 322668	
	− 6	83925	−258799	
	−16.66	64.606725	(7) 63869	
	− 6	83950	−58230	
	−16.72	(4) 64.690675	(8) 5639	
	− 6	3359	−5176	
	(3) −16.780	64.694034	(9) 463	
	− 5	3359	−453	
	−16.785	(5) 64.69739	(10) 10	
	− 5	118	−7	
	−16.790	64.69857	3	
	− 5	118		
	(4) −16.795	(6) 64.6997		
		1		
	(5) −16.8	64.6998		
		1		
	(6) −2	(7) 64.700		
		(8) 64.70		
		(9) 64.7		
		(10) 65.		

(420, Ex. 7)

8. Given, $X = x^4 + x^3 + x^2 - x - 500.$

Whence, $X_1 = 4x^3 + 3x^2 + 2x - 1.$

$$
\begin{array}{l|l}
4x^4 + 4x^3 + 4x^2 - 4x - 2000 & 4x^3 + 3x^2 + 2x - 1 \\
\underline{4x^4 + 3x^3 + 2x^2 - x} & \underline{x, \; +1} \\
x^3 + 2x^2 - 3x - 2000 & \\
\text{or,} \quad 4x^3 + 8x^2 - 12x - 8000 & \\
\underline{4x^3 + 3x^2 + 2x - 1} & \\
5x^2 - 14x - 7999. & R = -5x^2 + 14x + 7999.
\end{array}
$$

$$
\begin{array}{l|l}
20x^3 + 15x^2 + 10x - 5 & -5x^2 + 14x + 7999 \\
\underline{20x^3 - 56x^2 - 31996x} & \underline{-4x, \; -71} \\
71x^2 + 32006x - 5 & \\
\text{or,} \quad 355x^2 + 160030x - 25 & \\
\underline{355x^2 - 994x - 567929} & \\
161024x + 567904 &
\end{array}
$$

Whence, $R_1 = -5032x - 17747.$

Multiplying R by 5032,

$$
\begin{array}{l|l}
-25160x^2 + 70448x + 40250968 & -5032x - 17747 \\
\underline{-25160x^2 - 88735x} & \underline{5x, \; -159183} \\
159183x + 40250968 & \\
\text{or,} \quad 5032 \cdot 159183x + 202542870976 \cdot & \\
\underline{5032 \cdot 159183x + 2825020701} & \\
+199717850275 = -R_2 &
\end{array}
$$

In these functions, let

$x = -\infty$;	we have	+	−	−	+	−,	3 variations.
$x = +\infty$;	"	+	+	−	−	−,	1 variation.
$x = 0$;	"	−	−	+	−	−,	2 variations.

Hence, the given equation has one real root between 0 and ∞, which must be positive; and one real root between 0 and $-\infty$, which must be negative. By proper substitutions, we shall find the initial figures to be 4 and -4.

(420, Ex. 8)

The decimal part of the positive root is found as follows:

+1	+ 1	− 1	−500	4.4604168201
4	20	84	332	
5	21	83	(1) −168.0000	
4	36	228	142.9536	
9	57	(1) 311.000	(2) −25.04640000	
4	52	46.384	24.87028656	
13	(1) 109.00	357.384	(3) −.17611344	
4	6.96	49.232	.16900578	
(1) 17.0	115.96	(2) 406.616000	(4) −710766	
.4	7.12	7.888776	422569	
17.4	123.08	414.504776	(5) −288197	
.4	7.28	7.956168	253543	
17.8	(2) 130.3600	(3) 422.460944	(6) −34654	
.4	1.1196	53495	33806	
18.2	131.4796	422.514439	(7) −848	
.4	1.1232	53495	845	
(2) 18.60	132.6028	(4) 422.5679	(8) −3	
.06	1.1268	13	4	
18.66	(3) 133.7296	422.5692		
.06	75	13		
18.72	133.7271	(5) 422.571		
.06	75	(6) 422.57		
18.78	133.7446	(7) 422.6		
.06	75	(8) 423		
(3) 18.84	(4) 134.			

NOTE.—The first contracted terms in the operation, marked (4), occur in connection with the cipher in the root; and the pupil will observe that they are therefore contracted twice the usual number of places.

To obtain the decimal part of the negative root, change the signs of the alternate terms, and proceed as in the following operation :

−1	+ 1	+ 1	−500	4.9296646474
4	12	52	212	
3	13	53	(1) −288.0000	
4	28	164	275.7411	
7	41	(1) 217.000	(2) −12.25890000	
4	44	89.379	8.23961296	
11	(1) 85.00	306.379	(3) −4.01928704	
4	14.31	102.987	3.74208814	
(1) 15.0	99.31	(2) 409.366000	(4) −.27919890	
.9	15.12	2.614648	.25023185	
15.9	114.43	411.980648	(5) −2696705	
.9	15.93	2.622104	2502841	
16.8	(2) 130.3600	(3) 414.602752	(6) −193864	
.9	.3724	1.184819	166859	
17.7	130.7324	415.787571	(7) −27005	
.9	.3728	1.186331	25029	
(2) 18.60	131.1052	(4) 416.97390	(8) −1976	
2	.3732	7919	1669	
18.62	(3) 131.4784	417.05309	(9) −307	
2	.1681	7920	292	
18.64	131.6465	(5) 417.1323	(10) −17	
2	1681	79	17	
18.66	131.8146	417.1402	0	
2	.1681	79		
(3) 18.68	(4) 131.98	(6) 417.148		
(4) 2	1	(7) 417.15		
	131.99	(8) 417.2		
	1	(9) 417.	Hence, −4.9296646474, root.	
	132.00	(10) 42		
	1			
	(5) 132.			
	(6) 1			

(420, Ex. 8)

9. We have given

$$X = x^4 - 9x^3 - 11x^2 - 20x + 4.$$

The first derived polynomial is

$$X_1 = 4x^3 - 27x^2 - 22x - 20.$$

Multiplying X by 4, to avoid fractions in dividing,

$$\begin{array}{lrrrrr|l}
 & 4x^4 - 36x^3 & - \; 44x^2 & - \; 80x & + \; 16 & & 4x^3 - 27x^2 - 22x - 20 \\
 & 4x^4 - 27x^3 & - \; 22x^2 & - \; 20x & & & \hline x, \; -9 \\
\hline
 & - \; 9x^3 & - \; 22x^2 & - \; 60x & + \; 16 & & \\
\text{or} & -36x^3 & - \; 88x^2 & - 240x & + \; 64 & & \\
 & -36x^3 & + 243x^2 & + 198x & + 180 & & \\
\hline
 & & -331x^2 & - 438x & - 116. & & R = 331x^2 + 438x + 116.
\end{array}$$

Multiplying X_1 by 331,

$$\begin{array}{lrrrr|l}
 & 1324x^3 - & 8937x^2 - & 7282x - & 6620 & 331x^2 + 438x + 116 \\
 & 1324x^3 + & 1752x^2 + & 464x - & & \hline 4x, -10689 \\
\hline
 & & -10689x^2 - & 7746x - & 6620 & \\
\text{or} & & -331 \cdot 10689x^2 - & 2563926x - & 2191220, & \text{prepared dividend,} \\
 & & -331 \cdot 10689x^2 - & 4681782x - & 1239924 & \\
\hline
 & & & +2117866x - & 951296 &
\end{array}$$

Dividing by 32, and changing signs,

$$R_1 = -66183x + 29728.$$

Multiplying R by 66183,

$$\begin{array}{lrrr|l}
 & 66183 \cdot 331x^2 + & 28988154x + & 7677228 & -66183x + 29728 \\
 & 66183 \cdot 331x^2 - & 9839968x + & & \hline -331x, -19414061 \\
\hline
 & & +38828122x + & 7677228 & \\
\text{or,} & & +19414061x + & 3838614 & \\
\text{or,} & & +66183 \cdot 19414061x + & 254050990362, & \text{prepared dividend,} \\
 & & +66183 \cdot 19414061x - & 577141205408 & \\
\hline
 & & & +831192195770 = -R_2. &
\end{array}$$

(420, Ex. 9)

Thus we have, for the several functions,

$$\begin{aligned}
X &= x^4-9x^3-11x^2-20x+4,\\
X_1 &= 4x^3-27x^2-22x-20,\\
R &= 331x^2+438x+116,\\
R_1 &= -66183x+29728,\\
R_2 &= -831192195770.
\end{aligned}$$

To find the number of real roots, let

$x=-\infty$; signs,	+	−	+	+	−,	3 variations.
$x=+\infty$; "	+	+	+	−	−,	1 variation.
$x=0$; "	+	−	+	+	−,	3 variations.

Hence, there are two real roets between 0 and ∞. To find their situation, let

$x=0$; signs,	+	−	+	+	−,	3 variations.
$x=1$; "	−	−	+	−	−,	2 "
$x=2$; "	−	−	+	−	−,	2 "
$x=3$; "	−	−	+	−	−,	2 "
$x=4$; "	−	−	+	−	−,	2 "
$x=5$; "	−	−	+	−	−,	2 "
$x=6$; "	−	−	+	−	−,	2 "
$x=7$; "	−	−	+	−	−,	2 "
$x=8$; "	−	+	+	−	−,	2 "
$x=9$; "	−	+	+	−	−,	2 "
$x=10$; "	−	+	+	−	−,	2 "
$x=11$; "	+	+	+	−	−,	1 variation.

Hence the lesser root is situated between 0 and 1, and the greater between 10 and 11.

The initial figures of the greater are 10. To find the initial figure of the lesser, let

$x=.1$; signs,	+	−	+	−	−,	3 variations.
$x=.2$; "	−	−	+	−	−,	2 "

Hence, the initial figure is .1. The decimal part of this root is as follows:

(420, Ex. 9)

1 −9.0	−11.	−20.	+4 |.17968402504
.1	− .89	− 1.189	−2.1189
−8.9	−11.89	−21.189	(1) 1.88110000
.1	− .88	− 1.277	−1.64238179
−8.8	−12.77	(1) −22.466000	(2) .238718210000
.1	− .87	− .996597	−.221760635319
−8.7	(1) −13.6400	−23.462597	(3) 16957574681
.1	− .5971	− 1.038051	−14873731847
(1) −8.60	−14.2371	(2) −24.500648000	(4) 2083842834
7	− .5922	− .139422591	−1984015679
−8.53	−14.8293	−24.640070591	(5) 99827155
7	− .5873	− .140095053	−99206044
−8.46	(2) −15.416600	(3) −24.780165644	(6) 621111
7	− .74799	− 9387434	−496031
−8.39	−15.491399	−24.789553078	(7) 125080
7	− .74718	− 9390416	−124008
(2) −8.320	−15.566117	(4) −24.79894349	(8) 1072
9	− .74637	− 125250	−992
−8.311	(3) −15.640754	−24.80019599	80
9	− .4970	− 125255	
−8.302	−15.645724	(5) −24.8014485	
9	− .4970	− 626	
−8.293	−15.650694	−24.8016111	
9	− .4970	− 626	
(3) −8.284	(4) −15.6557	(6) −24.801574	
(4) −8.	− 6	(7) −24.8016	
	−15.6563	(8) −24.802	
	− 6		
	−15.6569		
	− 6		
	(5) −15.66		

NOTE—The term marked (6), in the second column from the right, is contracted *two places*, to obtain the next term (7). This is on account of the *cipher* which occurs in the corresponding part of the root.

(**420**, Ex. 9)

The decimal part of the greater root is found as follows:

−9	−11	−20	+ 4 | 10.2586086356
10	10	−10	−300
1	−1	−30	(1) −296.0000
10	110	1090	225.0096
11	109	(1) 1060.000	(2) −70.99040000
10	210	65.048	60.41618125
21	(1) 319.00	1125.048	(3) −10.57421875
10	6.24	66.304	9.82494438
(1) 31.0	325.24	(2) 1191.352000	(4) −.74927437
.2	6.28	16.971625	.73864151
31.2	331.52	1208.323625	(5) −1063286
.2	6.32	17.051375	985022
31.4	(2) 337.8400	(3) 1225.375000	(6) −78264
.2	1.5925	2.743048	73877
31.6	339.4325	1228.118048	(7) −4387
.2	1.5950	2.745096	3694
(2) 31.80	341.0275	(4) 1230.86314	(8) −693
5	1.5975	.20605	616
31.85	(3) 342.6250	1231.06919	(9) −77
5	.2560	.20606	74
31.90	342.8810	(5) 1231.2753	−3
5	.2560	27	
31.95	343.1370	1231.2780	
5	.2560	27	
(3) 32.00	(4) 343.39	(6) 1231.28	
	2		
(4) 3	343.41	(7) 1231.3	
	2		
	343.43	(8) 1231	
	2		
	(5) 343	(9) 123	

(420, Ex. 9)

10. We readily find the functions to be

$$X = x^4 - 12x^2 + 12x - 3,$$
$$X_1 = x^3 - 6x + 3,$$
$$R = 2x^2 - 3x + 1,$$
$$R_1 = 17x - 9,$$
$$R_2 = 8.$$

Let $x = -\infty$; we have	+	−	+	−	+,	3 variations.
" $x = +\infty$; "	+	+	+	+	+,	0 "

Hence the roots are all real. And since the signs of the given equation give three variations and one permanance, three of the roots are positive, and one is negative, (**447**).

To find the situation of the roots,

Let $x = 0$; we have	−	+	+	−	+,	3 variations.
" $x = 1$; "	−	−	±	+	+,	1 variation.
" $x = 2$; "	−	−	+	+	+,	1 "
" $x = 3$; "	+	+	+	+	+,	0 "
" $x = -1$; "	−	+	+	−	+,	3 variations.
" $x = -2$; "	−	+	+	−	+,	3 "
" $x = -3$; "	−	−	+	−	+,	3 "
" $x = -4$; "	+	−	+	−	+,	4 "

Hence, there are two roots situated between 0 and 1, one between 2 and 3, and one between −3 and −4.

To find more definitely the situation of the roots between 0 and 1.

Let $x = .1$; we have	−	+	+	−	+,	3 variations.
" $x = .2$; "	−	+	+	−	+,	3 "
" $x = .3$; "	−	+	+	−	+,	3 "
" $x = .4$; "	−	+	+	−	+,	3 "
" $x = .5$; "	+	+	±	−	+,	2 "
" $x = .6$; "	+	−	−	+	+,	2 "
" $x = .7$; "	−	−	−	+	+,	1 variation.

Hence, there is one root between .4 and .5, and one between .6 and .7.

The initial figures of the four roots, taken in the order of their values, are as follows:

2, .6, .4, −3.

(**420**, Ex. 10)

The decimal part of the first root is found as follows:

1+0	−12	+12	−8	2.858083082
2	4	−16	−8	
2	− 8	− 4	(1)−11.0000	
2	8	0	8.9856	
4	0	(1)− 4.000	(2)−2.01440000	
2	12	15.232	1.71940625	
6	(1)+12.00	+11.232	(3)−.29499375	
2	7.04	21.376	.29192888	
(1) 8.0	19.04	(2) 32.608000	(4)−306487	
.8	7.68	1.780125	294315	
8.8	26.72	34.388125	(5)−12172	
.8	8.32	1.808375	11038	
9.6	(2) 35.0400	(3) 36.196500	(6)−1134	
.8	.5625	.294610	1104	
10.4	35.6025	36.491110	(7)−30	
.8	.5650	.295340	29	
(2) 11.20	36.1675	(4) 36.78645	(8)−1	
5	.5675	296	1	
11.25	(3) 36.7350	36.78941	0	
5	912	296		
11.30	36.8262	(5) 36.792		
5	912	(6) 36.79		
11.35	36.9174	(7) 36.8		
5	912	(8) 4		
(3) 11.40	(4) 37.01			
(4) 1				

(420, Ex. 10)

The second root is found as follows:

1	+0.	−12.	+12	−3 |.6060183069
	.6	+ .36	− 6.984	3.0096
	.6	−11.68	+5.016	(1)+9600000000
	.6	+ .72	-6.552	−9569720304
	1.2	−10.92	(1)−1.536000000	(2)+30279696
	.6	+ 1.08	− 58953384	−16539179
	1.8	(1)−9.840000	−1.594953384	(3)+13740517
	6	+ 14436	− 58866552	−13232754
	(1) 2.400	−9.825564	(2)−1.653819936	(4)+507763
	6	+ 14472	− 97996	−496253
	2.406	−9.811092	−1.653917902	(5)+11510
	6	+ 14508	− 97965	− 9925
	2.412	(2)−9.796584	(3)−1.6540059	(6)+1585
	6	+ 24	− 784	−1489
	2.418	−9.796560	−1.6540943	+96
	6	+ 24	− 784	
	(2) 2.424	−9.796536	(4)−1.654173	
		24	− 3	
		(3)−9.80	−1.654176	
			3	
		(4)−10	(5)−1.65418	
			(6) 1.654	

(420, Ex. 10)

The third root is found as follows:

1	+0	−12.	+12	−3 |.4432769396
	.4	.16	− 4.736	2.9056
	.4	−11.84	7.264	(1)−.09440000
	.4	.32	−4.308	8864096
	.8	−11.52	(1) 2.656000	(2)−5719040000
	.4	.48	−.438976	5244710001
	1.2	(1)−11.0400	2.217024	(3)−474329999
	.4	656	−.436288	342717760
(1)	1.60	−10.9744	(2) 1.780736000	(4)−131612239
	4	672	−32499333	119746687
	1.64	−10.9072	1.748236667	(5)−11865552
	4	688	−32483439	10259068
	1.68	(2)−10.838400	(3) 1.715753228	(6)−1606484
	4	5289	−2164430	1538793
	1.72	−10.833111	1.713588798	(7)−67691
	4	5298	−2164360	51293
(2)	1.760	−10.827813	(4) 1.71142444	(8)−16398
	3	5307	−75749	15388
	1.763	(3)−10.822506	1.71076695	(9)−1010
	3	354	−75748	1026
	1.766	−10.822152	(5) 1.7099095	
	3	354	−649	
	1.769	−10.821798	1.7098446	
	3	354	−649	
(3)	1.772	(4)−10.8214	(6) 1.709780	
		1	−10	
(4)	2	−10.8213	1.709770	
		1	−10	
		−10.8212	(7) 1.60976	
		1		
		(5)−10.82	(8) 1.7098	
		(6)−11	(9) 1.710	

(490, Ex. 10)

The decimal part of the fourth root is found as follows, the alternate signs in the given equation being changed:

1	−0	−12	−12	− 3	3.9073785547
	3	9	− 9	−63	
	3	−3	−21	(1) −66.0000	
	3	18	45	65.0241	
	6	15	(1) 24.000	(2) −.97590000	
	3	27	48.249	.92562109	
	9	(1) 42.00	72.249	(3) −5027891	
	3	11.61	59.427	3984354	
	(1) 12.0	53.61	(2) 131.676000	(4) −1043537	
	.9	12.42	.555584	929889	
	12.9	66.03	132.231584	(5) −113648	
	.9	13.23	.556349	106278	
	13.8	(2) 79.2600	(3) 132.78793	(6) −7370	
	.9	.1092	2388	6643	
	14.7	79.3692	132.81181	(7) −727	
	.9	.1092	2388	665	
	(2) 15.60	79.4784	(4) 132.8357	(8) −62	
		.1092	56	53	
	(3) 2	(3) 79.59	132.8413	(9) −9	
			56	9	
		(4) 80.	(5) 132.847	0	
			(6) 132.85		
			(7) 132.9		
			(8) 133.		
			(9) 13.	−3. 9073785547, root.	

(420, Ex. 10)

11. We have given,

$$X = x^5 - 10x^3 + 6x + 1.$$

The first derived polynomial is

$$X_1 = 5x^4 - 30x^2 + 6.$$

Multiplying X by 5, to avoid fractions,

$$\begin{array}{r|l}
5x^5 - 50x^3 + 30x + 5 & 5x^4 - 30x^2 + 6 \\ \hline
5x^5 - 30x^3 + 6x & x \\ \hline
-20x^3 + 24x + 5. &
\end{array}$$

Hence, $\qquad R = 20x^3 - 24x - 5.$

Multiplying X_1 by 4, to avoid fractions,

$$\begin{array}{r|l}
20x^4 - 120x^2 + 24 & 20x^3 - 24x - 5 \\ \hline
20x^4 - 24x^2 - 5x & x \\ \hline
-96x^2 + 5x + 24 &
\end{array}$$

Hence, $\qquad R_1 = 96x^2 - 5x - 24.$

Multiplying R by 24,

$$\begin{array}{r|l}
480x^3 - 576x - 120 & 96x^2 - 5x - 24 \\ \hline
480x^3 - 25x^2 - 120x & 5x, +25 \\ \hline
25x^2 - 456x - 120; & \text{multiply by } 96 \\
2400x^2 - 43776x - 11520 & \\
2400x^2 - 125x - 600 & \\ \hline
-43651x - 10920. &
\end{array}$$

Hence, $\qquad R_2 = 43651x + 10920.$

Multiplying R_1 by 43651,

$$\begin{array}{r|l}
4190496x^2 - 218255x - 1047624 & 43651x + 10920 \\ \hline
4190496x^2 + 1048320x & 96x, -1266575 \\ \hline
-1266575x - 1047624; & \text{multiply by } 43651 \\
-55287265325x - 45729835224 & \\
-55287265325x - 13830999000 & \\ \hline
-31898836224 = -R_3. &
\end{array}$$

Therefore, the functions are

$$X = x^5 - 10x^3 + 6x - 1, \qquad X_1 = 5x^4 - 30x^2 + 6,$$
$$R = 20x^3 - 24x - 5, \qquad R_1 = 96x^2 - 5x - 24,$$
$$R_2 = 43651x + 10920, \qquad R_3 = 31898836224.$$

Let $x = -\infty$; we have $\quad - \quad + \quad - \quad + \quad - \quad +,\quad$ 5 variations.

" $x = +\infty$; " $\quad + \quad + \quad + \quad + \quad + \quad +,\quad$ 0 "

(420, Ex. 11)

Hence, there are 5 real roots. And since the signs of the given equation present two variations, two of the roots are positive, (**447**); consequently, three are negative.

To ascertain the situation of the roots, let

$x=0$;	signs,	+	+	−	−	+	+,	2	variations.
$x=1$;	"	−	−	−	+	+	+,	1	variation.
$x=2$;	"	−	−	+	+	+	+,	1	"
$x=3$;	"	−	+	+	+	+	+,	1	"
$x=4$;	"	+	+	+	+	+	+,	0	"
$x=-1$;	"	+	−	−	+	−	+,	4	variations.
$x=-2$;	"	+	−	−	+	−	+,	4	"
$x=-3$;	"	+	+	−	+	−	+,	4	"
$x=-4$;	"	−	+	−	+	−	+,	5	"

Hence we have

1 positive root between	0 and	1,
1 " " "	3 and	4,
2 negative roots "	0 and	−1,
1 " root "	−3 and	−4.

In order to limit still further the roots situated between 0 and 1, and 0 and −1, we may employ X alone, according to (**452, 3**).

As $x=1$ reduces X nearly to 0, the positive root between 0 and 1 must be nearly 1. We therefore commence with $x=1$, and substitute the descending scale of tenths, till we come to the initial figure. Thus,

Substitute $x=$	1,	.9,	.8,
The signs of X are	−	−	+

Hence, this root is situated between .8 and .9.

For the roots situated between 0 and −1, we proceed thus:

Substitute $x=$	0,	−.1,	−.2,	−.3,	−.4,	−.5,	−.6,	−.7,
The signs of X are	+	+	−	−	−	−	−	+

Hence, one root is between −.1 and −.2; and the other between −.6 and −.7.

Thus, we find the *initial figures* of the several roots, taken in the order of their algebraic values, to be

$$-3, \quad -.6, \quad -.1, \quad +.8, \quad +3.$$

NOTE.—In extracting the negative roots in this example, we may change the signs of the alternate terms, remembering always to supply the deficient terms, with 0 for coefficients.

(**420**, Ex. 11)

The operation for the first root is as follows :

1 — 0	— 10	— 0	+ 6	— 1 | 3.0653157913
3	9	— 3	— 9	— 9
3	— 1	— 3	— 3	(1) — 10.0000000000
3	18	51	144	9.1254751776
6	17	48	141.00000000	(2) — .8745248224
3	27	132	11.09125296	.8222637421
9	44	(1) 180.000000	152.09125296	(3) — 522610803
3	36	4.854216	11.38577184	496468108
12	(1) 80.0000	184.854216	(2) 163.47702480	(4) — 26142695
3	.9036	4.908648	.97572362	16555014
(1) 15.00	80.9036	189.762864	164.45274842	(5) — 9587681
6	.9072	4.963296	.97781834	8277654
15.06	81.8108	(2) 194.726160	(3) 165.4305668	(6) — 1310027
6	.9108	.418563	588025	1158879
15.12	82.7216	195.144723	165.4893693	(7) — 151148
6	.9144	.418945	588100	148999
15.18	(3) 83.6360	195.563668	(4) 165.548179	(8) — 2149
6	765	.419328	1961	1656
15.24	83.7125	(3) 195.9830	165.550140	(9) — 493
6	765	252	1961	497
(2) 15.30	83.7890	196.0082	(5) 165.55210	+ 4
	765	252	98	
	83.8655	196.0334	165.55308	
	765	252	98	
	(3) 83.9	(4) 196.06	(6) 165.5541	
		(5) 196.	1	
		(6) 2	165.5542	
			(7) 165.554	
			(8) 165.55	
			(9) 165.6	— 3.0653157913, root.

(420, Ex. 11)

The operation for the second root is as follows:

1 − 0	−10	− 0	+6	−1|.6915752805
.6	.36	− 5.784	−3.4704	1.51776
.6	−9.64	− 5.784	+2.5296	(1)+.5177600000
.6	.72	− 5.352	−6.6816	−.5064468651
1.2	−8.92	−11.136	(1)−4.15200000	(2) 113131349
.6	1.08	− 4.704	−1.47518739	−71670641
1.8	−7.84	(1)−15.840000	−5.62718739	(3) 41460708
.6	1.44	− .550971	−1.52245656	−35966031
2.4	(1)−6.4000	−16.390971	(2)−7.14964395	(4) 5494677
.6	.2781	− .525213	− 1742015	−5042202
(1) 3.00	−6.1219	−16.916184	−7.16706410	(5) 452475
9	.2862	− .498726	− 1742538	−432268
3.09	−5.8357	(2)−17.414910	(3)−7.1844895	(6) 20207
9	2943	− 5236	− 87166	−14409
3.18	−5.5414	−17.420146	−7.1932061	(7) 5798
9	.3024	− 5232	− 87179	−5764
3.27	(2)−5.2390	−17.425378	(4)−7.201924	(8) 34
9	35	− 5229	− 1221	−36
3.36	−5.2355	(2)−17.4306	−7.203145	−2
9	35	− 26	− 1221	
(2) 3.45	−5.2320	−17.4332	(5)−7.20437	
	35	− 26	− 10	
	−5.2285	−17.4358	−7.20447	
	35	− 26	− 10	
	−5.2	(4)−17.44	(6)−7.2046	
		(5)−17.	(7)−7.205	
			(8)−7.21	−.6915762805, root.

(420, Ex. 11)

The operation for the third root is as follows:

1− 0	−10	−0	+6.	−1.|.1756747993
.1	.01	− .999	− .0999	.59001
.1	−9.99	− .999	5.9001	(1)−.4099900000
.1	2	− .997	− .1996	.3810019857
.2	−9.97	−1.996	(1) 5.70050000	(2)−289880143
.1	3	− .994	− .25761449	255583952
.3	−9.94	(1)−2.990000	5.44288551	(3)−34296191
.1	4	− .690207	− .30570946	30496909
.4	(1)−9.9000	−3.680207	(2) 5.13717605	(4)−3799282
.1	399	− .687071	− 2549702	3555530
(1) .50	−9.8601	−4.367278	5.11167903	(5)−243752
7	448	− .683592	− 2573958	203158
.57	−9.8153	(2)−5.050870	(3) 5.0859394	(6)−40594
7	497	− 48534	− 31213	35552
.64	−9.7656	−5.099404	5.0828181	(7)−5042
7	546	− 48512	− 31248	4571
.71	(2)−9.7110	−5.147916	(4) 5.079693	(8)−471
7	43	− 48491	− 363	457
.78	−9.7067	(3)−5.1964	5.079328	(9)−14
7	43	− 58	− 365	15
(2) .85	−9.7024	−5.2022	(5) 5.07896	+1
	43	− 58	− 2	
	−9.6981	−5.2080	−5.07894	
	43	− 58	− 2	
	(3)−9.7	(4)−5.21	(6) 5.0789	
		(5)−5.	(7) 5.079	
			(8) 5.08	
			(9) 5.1	−.1756747993, root.

(420, Ex. 11)

The operation for the fourth root is as follows:

1 +0	−10.	+0	+6	+1 |.8795087084
.8	.64	−7.488	−5.9904	+ .00768
.8	−9.36	−7.488	+ .0096	(1) 1.0076800000
.8	1.28	−6.464	−11.1616	−.8742890793
1.6	−8.08	−13.952	(1)−11.15200000	(2).1333909207
.8	1.92	− 4.928	− 1.38784399	−.1261650637
2.4	−6.16	(1)−18.880000	−12.48984399	(3) 72258570
8	2.56	− .232057	− 1.35266796	−71020747
3.2	(1)−3.6000	−19.112057	(2)−13.84251195	(4) 1237823
.8	.2849	− .211771	− .17582846	−1137128
(1) 4.00	−3.3151	−19.323828	−14.01834041	(5) 100695
7	.2898	− .191142	− .17601901	−99500
4.07	−3.0253	(2)−19.514970	(3)−14.1943594	(6) 1195
7	.2947	− 21525	− 97899	−1137
4.14	−2.7306	−19.536495	−14.2041493	(7) 58
7	.2996	− 21173	− 97905	−57
4.21	(2)−2.4310	−19.557668	(4)−14.213940	1
7	392	− 20821	− 157	
4.28	−2.3918	(3)−19.5785	−14.214097	
7	392	− 12	− 157	
(2) 4.35	−2.3526	−19.5797	(5)−14.2143	
	392	− 12		
	−2.3134	−19.5809	(6)−14.214	
	392	− 12		
	(3)−2.3	(4)−19.58	(7)−14.2	

(420, Ex. 11)

The operation for the fifth root is as follows:

1+0	−10	+0	+6	+1	3.0535816266
3	9	−3	−9	−9	
3	−1	−3	−3	(1) −8.0000000000	
3	18	51	144	7.5100940625	
6	17	48	(1) 141.00000000	(2) −.4899059375	
3	27	132	9.20188125	.4805548729	
9	44	(1) 180.000000	150.20188125	(3) −93510646	
3	36	4.037625	9.40565000	80386391	
12	(1) 80.0000	184.037625	(2) 159.60753125	(4) −13124255	
3	.7525	4.075375	.57742639	12862718	
(1) 15.00	80.7525	188.113000	160.18495764	(5) −261537	
5	.7550	4.113250	.57817444	160786	
15.05	81.5075	(2) 192.226250	(3) 160.7631321	(6) −100751	
5	.7575	.249212	96489	96472	
15.10	82.2650	192.475462	160.7727810	(7) −4279	
5	.7600	.249350	96491	3216	
15.15	(3) 83.0250	192.724812	(4) 160.78243	(8) −1063	
5	458	.249487	154	965	
15.20	83.0708	(3) 192.9743	160.78397	(9) −98	
5	458	42	154	97	
(2) 15.25	83.1166	192.9785	(5) 160.7855	−1	
	458	42	(6) 160.786		
	83.1624	192.9827	(7) 160.79		
	458	42	(8) 160.8		
	(3) 83.2	(4) 193.	(9) 161.		

(420, Ex. 11)